Title	STUDIES ON TRANSFORMATIONS FOR NONLINEAR FIELD EQUATIONS WITH SYMMETRY(Dissertation_全文)
Author(s)	Nakamura, Yoshimasa
Citation	Kyoto University (京都大学)
Issue Date	1983-09-24
URL	http://dx.doi.org/10.14989/doctor.k3001
Right	
Type	Thesis or Dissertation
Textversion	author

STUDIES ON TRANSFORMATIONS FOR NONLINEAR FIELD EQUATIONS WITH SYMMETRY

YOSHIMASA NAKAMURA

1983

STUDIES ON TRANSFORMATIONS FOR NONLINEAR FIELD EQUATIONS WITH SYMMETRY

Yoshimasa NAKAMURA

submitted in partial fulfilment of
the requirements for the degree of
DOCTOR OF ENGINEERING
(Applied Mathematics and Physics)
at
Kyoto University

MAY 1983

CONTENTS

CHAPTER I

INTRODUCTION

There has been a long tradition of mutual contact between differential geometry and mathematical physics. Some of physical theories of historical importance indeed owe their fundamental ideas to differential geometry. As a widely known example, the general relativity originated by A. Einstein exploits the tensor analysis developed in Riemannian geometry. According to the theory, the metric tensor which describes the vacuum gravitational fields is required to satisfy the Einstein equations $R_{\mu\nu} = 0$, tensor equations. On the other hand, the non-abelian gauge theory originated by C. N. Yang and his coworkers is closely related to the geometry of fiber bundles. Each solutions of the Yang-Mills equations $D_\mu F_{\mu\nu} = 0$, which are the fundamental equations in gauge fields, can be interpreted as a harmonic connection 1-form on a principal fiber bundle. The notion of nonabelian gauge fields is a natural generalization of Maxwell's electromagnetic fields. Complete understanding of these field equations should contribute back to further progress in differential geometry.

Exact solutions of the field equations, if they are found, provide detailed information in the understanding of the fields. However, since both the Einstein equations and the Yang-Mills equations are nonlinear and rather complicated, there has been no systematic method for finding exact solutions. Reduction to linear equations under the assumption of suitable symmetries or lower dimensional sectors therefore have been developed to yield several important special solutions. For example, it is shown that the Einstein equations admit the Schwarzchild-Weyl series of static axially symmetric solutions [24]. As for gauge theory, some exact solutions are found such as Ikeda-Miyachi's static spherically symmetric solutions [17, 18, 31] and the instanton solutions [3] of the Yang-Mills equations.

Matters being so, it seems of great use to develop a method of finding new solutions from the old, which generates a series of exact solutions systematically. The method, rather purely mathematical, is worth studying.

In recent years remarkable prograss has been made in the study of nonlinear differential equations. Three fruitful approaches have been proposed independently. The first is *the soliton theory* which includes the inverse scattering method, the Bäcklund transformations, and the Riemann-Hilbert problem. The second is to make extensive use of *the symmetries* which the nonlinear equations admit. The third is called *the twistor theory* applied to nonlinear fields in four-dimensional space-times. These approaches are to be described below.

A start is made with the soliton theory. In 1967, Gardner-Green-Kruskal-Miura [11] found that the Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$ in hydrodynamics can be solved exactly. A class of particular solutions called *the solitons* was obtained by using a scattering theory for the one-dimensional Schrödinger equation. Their technique is referred to as *the inverse scattering method.* Here the name "soliton" solutions comes from the property of the solutions that they take the form of localized disturbances which retain their shapes after an interaction among themselves. It is also to be noted that the Korteweg-de Vries equation possesses an infinite number of local conservation laws [30] leading to independent integrals of motion which are in involution [42]. This proves the complete integrability of the equation. Furthermore, the inverse scattering method was extended to other two-dimensional nonlinear differential equations in a manner such that the nonlinear equations are equivalent to the compatibility conditions for certain linear differential equations which are called *the inverse scattering formulae* [44].

The sine-Gordon equation $\omega_{\xi\eta} = \sin \omega$ goes back to the nineteenth century when this equation emerged in the differential geometry of surfaces. To each solution of the equation there corresponds a surface of constant negative Gaussian curvature [9]. Soliton solutions of the sine-

Gordon equation are obtained by the method of *Bäcklund transformation* which relates the old solutions to new ones. In addition multi-soliton solutions can be constructed from a single soliton solution through purely algebraic means. The construction is guaranteed by the theorem of permutability [25] of the Bäcklund transformations.

Although the origin of the sine-Gordon equation is in differential geometry, it is now viewed as a starting point of the soliton theory for nonlinear field equations which appear in mathematical physics. Pohlmeyer [34] showed that the relativistic invariant nonlinear $O(3)$ sigma-models $q_{\xi\eta} + (q_\xi, q_\eta)q = 0$, $q \in S^2$ are reduced to the sine-Gordon equation under a suitable dependent variable transformation. It is known that the $O(3)$ sigma-models can be written in the form $\partial_\mu(\partial_\mu g \cdot g^{-1}) = 0$, $g \in SU(2)$, which are called the $SU(2)$ chiral field equations. Other G-valued chiral fields $g(x)$ have been discussed by many authors, where G is a Lie group or its quotient space such as $\mathbb{P}^n$, a projective space. For example, the $SU(3)$ chiral field equations are interpreted as the Gauss-Codazzi equations of a surface embedded in S^4 [27]. It is worth noticing that the sine-Gordon equation admits an infinite number of local conservation laws, while the chiral field equations possess an infinite set of nonlocal conservation laws [28].

The method of *Riemann-Hilbert problem* in the theory of ordinary differential equations is of practical use in integrating the chiral field equations [43]. It should be remarked that the Riemann-Hilbert problem can be solved algebraically to yield soliton-type solutions. This technique seems to be applicable in the study of soliton theory of other nonlinear field equations.

The soliton theory of the chiral field equations gives rise to investigations of certain kind of nonlinear field equations. For example, if the background field is constant, the stationary axially symmetric vacuum Einstein equations are reduced to the $O(2,1)$ nonlinear sigma-models. Though this seems to have no physical grounds, it suggests that the soliton theory will offer an effective tool for research of exact solutions in the gravitational field theory. Indeed, inverse scattering formulae for gravitational fields have been found in [5, 29], and Bäcklund transformations in [13, 32]. However, no concrete solutions have been derived in these works.

Besides the above methods, there is the second approach to generate exact solutions of nonlinear field equations, which has been developed in the last decade. It makes essential use of the symmetry of field equations in the construction of new solutions from the old. Here the terminology symmetry means invariance under certain transformations. Such a way of obtaining new solutions has its origin in the study of stationary Einstein's gravitational fields. Using the Ehlers $SL(2, \mathbb{R})$ rotation and the covariance under the coordinate transformations, Geroch [12] showed that an infinite-dimensional transformation group called *the Geroch group* is admitted by the stationary axially symmetric vacuum Einstein equations. Roughly speaking, this group is a free product group of two finite-dimensional symmetry groups. He also conjectured that the Geroch group acts transitively on the solution space. Pursuing this approach further, Kinnersley and Chitre [20-23] have exponentiated several classes of the infinitesimal transformations of the Geroch group, and thereby succeeded in constructing the series of Kerr-Tomimatsu-Sato solutions. The algebra formed of the infinitesimal transformations, *a symmetry algebra*, is isomorphic to the graded Lie algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ of $sl(2, \mathbb{R})$-valued Laurent polynomials called *the affine Lie algebra* [26].

Recently, part of the Geroch conjecture has been proved affirmatively by Hauser and Ernst [16]. A class of stationary axially symmetric space-times can be generated from Minkowski's flat space-time by Kinnersley-Chitre's infinitesimal transformations. They also have formally exponentiated all of these transformations by using the

Riemann-Hilbert problem [14, 15]. However, there still remain the following problem. Only the infinitesimal transformations have been treated thus far. Is it possible to work out exact solutions by the use of finite transformations or Bäcklund transformations of the Geroch group? A deeper understanding of the Geroch group will be the key to finding general solutions of stationary Einstein's gravitational fields.

The third approach to solve nonlinear field equations is to use techniques in complex geometry. The Yang-Mills equations $D_\mu F_{\mu\nu} = 0$ require that the sum of covariant derivatives of the gauge field strengths vanishes. These are second-order differential equations for gauge potentials. The Bianchi identity $D_\mu {*F}_{\mu\nu} = 0$ then implies that a solution of the first-order differential equations ${*F}_{\mu\nu} = F_{\mu\nu}$ called the self-dual equations automatically satisfies the Yang-Mills equations. It is useful to treat the self-dual equations in $\mathbb{C}^4$ which comprises the ordinary Minkowski space and the Euclidean space. Then the self-dual equations take the form of four-dimensional chiral field equations. Along this line of thought, several attempts have been made in the framework of the soliton theory. The first of them is the discovery of inverse scattering formulae [4, 35] that yield an infinite number of nonlocal conservation laws [35, 37]. The second is concerned with Bäcklund transformations [4, 6, 35, 36]. These transformations construct no new solutions and violate the additional reality conditions. In spite of these works, the soliton theory has yet been unsatisfactory for gauge fields.

Apart from these works, complex analytic geometry have a marked impact on the gauge theory. For instance, a nice method for generating particular solutions of four-dimensional field equations such as instantons has been proposed from the viewpoint of *the twistor theory* developed by Penrose [33]. Further, Ward [40] found that all the information of the $SU(2)$ self-dual equations can be coded in a certain analytic vector bundle on $\mathbb{P}^3$. Atiyah and Ward [2] pointed out that the self-dual gauge potentials are obtained from a transition function between two coordinate patches of the vector bundle. Although the instanton problem in the gauge theory was solved in principle in [1, 2], their moduli space is still very limited. Soliton theory is then hoped to work in the study of non-instanton solutions. It should be noted that Atiyah-Ward's procedure is closely related to the Riemann-Hilbert problem in soliton theory.

The present article has two main themes;

(i) soliton theory; inverse scattering formulae, Bäcklund transformation, Riemann-Hilbert problems and exact solutions,

(ii) symmetry theory; transformation groups, symmetry algebras, conservation laws and Noether transformations.

These are concerned with integrable nonlinear field equations, mainly, stationary Einstein's gravitational field equations in the general relativity and the self-dual Yang-Mills equations in the gauge theory. New inverse scattering formula and Bäcklund transformations for the Einstein equations are given. It is shown that a finite transformation of the Geroch group yields new family of exact solutions. Three types of Bäcklund transformations for the self-dual equations are proposed from the viewpoint of the Riemann-Hilbert problem. Then the relationship between the twistor theory and the Riemann-Hilbert problem is discussed. It is also proved that an infinite-dimensional Lie algebra acts on the solution space of the self-dual equations. Infinitesimal transformations of the Geroch-like group in gauge fields are given explicitly. Finally, a method for finding Noether transformations for nonlinear fields is proposed.

The contents of the article are organised as follows.

Chapter II is devoted to the soliton theory of the chiral field equations. First, it is shown that the usual Bäcklund transformation of the sine-Gordon equation is connected to the method of the Riemann-Hilbert problem for solving the $SU(2)$ chiral field equations developed by Zakharov and Mikhailov [43]. An inverse scattering formula for the general chiral field equations is derived by the method of prolongation which was introduced by Wahlquist and Estabrook [39] for the Korteweg-de Vries equation. The inverse scattering formula is used to yield a set of non-local conservation laws of the chiral field equations.

Chapter III and Chapter IV are concerned with the soliton theory of Einstein's gravitational field equations. Two possible gravitational fields are known. One is the stationary axially symmetric vacuum fields. The other is the spherically symmetric vacuum fields. As there is no essential difference between the methods for solving them, we restrict ourselves to the former case which depends on (ρ, z), two of cylindrical coordinates. It is to be noticed that the famous Tomimatsu-Sato solutions were found via the Ernst form [10] of stationary Einstein's equations. In Chapter III, by defining a reduced form of the Ernst equation, the Phaff equations which vanish on the solution space are introduced. An inverse scattering formula is also discussed in terms of $SL(2,\mathbb{R})$ fiber bundle. Chapter IV deals with transformation theories for the Einstein equations, that is, Bäcklund transformations and Riemann-Hilbert transformations are proposed. An application is made to construct the Kasner-type solutions and their generalizations by the successive use of the Bäcklund transformations. A formalism for three-dimensional nonlinear equations whose solutions include all the solutions of the stationary axially symmetric equations is studied in the notion of the Riemann-Hilbert problem.

Chapter V considers symmetries of the stationary axially symmetric vacuum Einstein equations. First, Geroch's transformation group is reviewed. The $SL(2,\mathbb{R})$ rotation group and an internal symmetry are combined to construct a Bäcklund transformation in addition to that due to Kinnersley and Chitre. For a special class of exact solutions, the Bäcklund transformation is actually integrated to give a new family of exact solutions which are expressed by the ratio of determinants. Concrete metrics are also presented.

Chapter VI studies the soliton theory for the self-dual Yang-Mills equations in the gauge theory. Little is known of full integration method for the Yang-Mills equations. The self-dual Yang-Mills equations can be transformed into the so-called $SL(n,\mathbb{C})$ self-dual equations.

The key to solving the self-dual equations will be furnished by the method of the Riemann-Hilbert problem to be developed in Chapter IV. An inverse scattering formula which is slightly different from known ones is obtained in the framework of the prolongation. A transformation theorem is proved which gurantees that solutions of the self-dual equations can be derived by using the Riemann-Hilbert problem. Three types of Bäcklund transformations are analyzed explicitely. The first one gives the t' Hooft N-instanton solution having 5N arbitrary parameters. The second relates the Atiyah-Ward ansatz A_N to A_{N+1} of twistor theory. Lemmas in linear algebra give a expression to A_N. The last transformation keeps the reality conditions of $SU(n)$ gauge potentials.

In Chapter VII, symmetries of the self-dual Yang-Mills equations are discussed in terms of the $GL(n, \mathbb{C})$ self-dual equations. So far it has been resigned to prove the suspected complete integrability of gauge fields. If there is an infinite-dimensional transformation group such as the Geroch group in gravitational fields, the gauge field equations possess an infinite number of local and/or non-local conservation laws. A question then arises as to whether there exists a Geroch-like transformation group or not. It is shown for the $SL(2, \mathbb{C})$ self-dual equations that an Ehlers-type $SL(2, \mathbb{C})$ rotation group acts on a particular factrization, called the R-gauge [41], of the $SL(2, \mathbb{C})$-valued dependent variables. Combining this group with an internal symmetry gives rise to infinitesimal transformations of the above group. The results are more complicated than the Kinnersley-Chitre transformations of the Geroch group. An infinite number of nonlocal symmetries is obtained by the inverse scattering formula. The symmetry algebra of the self-dual equations is also discussed. The Riemann-Hilbert transformation studied in Chapter VI gives rise to an infinite-dimensional Lie algebra $gl(n, \mathbb{C}) \otimes \mathbb{C}[w, w^{-1}, w_1, w_2]$ which acts on the solution space of the $GL(n, \mathbb{C})$ self-dual equations.

Chapter VIII deals with the Noether transformations for nonlinear fields. A method is proposed for finding Noether transformations which yield weak continuity equations. It is worth noting that the equations of motion for the static Yang-Mills-Higgs fields allow of finite energy solutions called the monopoles [19]. However, nontrivial continuity equations and symmetries are known of to a small extent. It is shown that a large class of weak continuity equations exists in the Yang-Mills-Higgs fields as well as the chiral fields. These equations are derived from the Noether transformations as Noether currents. On the other hand, it is known that the chiral fields admit Noether transformations called the hidden symmetries [7, 8], which make change in the Lagrangian density by total divergences. The generators of the hidden symmetries form an affine Lie algebra $g \otimes \mathbb{C}[\zeta]$. It is shown that the hidden symmetry algebra is a subalgebra of the symmetry algebra $g \otimes \mathbb{C}[\zeta, \zeta^{-1}]$, given in [38], of the infinitesimal Riemann-Hilbert transformations.

The final chapter is concerned with concluding remarks. As this article will reveal, there is a fundamental relationship between the soliton theory and the symmetry theory. Furthermore, these two afford a close insight into Einstein's gravitational fields and the Yang-Mills gauge fields. There remains of course a number of problems to be solved. This chapter contains remarks about the relationship mentioned above and a further outlook in the study of nonlinear field equations.

CHAPTER II

SOLITON THEORY OF CHIRAL FIELD EQUATIONS

The sine-Gordon equation originally emerged in differential geometry of surfaces. As is well known, solutions of the sine-Gordon equation correspond to surfaces of constant negative curvature in $\mathbb{R}^3$. In 1882 Bäcklund found a transformation mapping such surfaces into themselves. His technique has recently received much attention in the study of soliton solutions to the sine-Gordon equation. For Bäcklund's transformation generates multi-soliton solutions from a trivial one. The sine-Gordon equation is now recognized to be one of the typical *soliton equations*.

On the other hand, there has been much interest in two-dimensional nonlinear chiral fields in field theory. This is due to the fact that they are in many respects analogous to the four-dimensional nonabelian gauge fields which are of great importance from a physical point of view but more difficult to study. Both have such properties as conformal invariance, asymptotic freedom, and admit topologically nontrivial solutions of equations of motion.

It should be noted that the $SU(2)$ chiral field equations are reduced to the celebrated sine-Gordon equation. This allows us to set up the soliton theory for the chiral fields, that is, to construct the Bäcklund transformation, the inverse scattering formula and conservation laws.

In Section 2.1, the sine-Gordon equation is derived from $O(3)$ nonlinear sigma-models by breaking their scale invariance. In Section 2.2, we build up the relationship between the usual Bäcklund transformation and the method of Riemann-Hilbert problem. In Section 2.3, an inverse scattering formula for the chiral field equations is proposed by using the method of prolongation. Section 2.4 presents nonlocal conservation laws of the chiral field equations.

2.1. Sine-Gordon equation and chiral field equations

Let us consider a surface of constant negative Gaussian curvature $K = -1/a^2$, a pseudosphere, embedded in a three-dimensional Euclidean space $\mathbb{R}^3$. Here a is a positive constant. It is always possible to choose the isothermal coodinates u and v such that the first fundamental form is expressed as

$$ds^2 = a^2\cos^2(\omega/2)du^2 + a^2\sin^2(\omega/2)dv^2.$$

The Gaussian curvature of the surface is then given by

$$K = -\frac{\omega_{uu} - \omega_{vv}}{2a^2\sin(\omega/2)\cos(\omega/2)},$$

where $\omega_{uu} = \partial_u{}^2\omega$, $\partial_u = \partial/\partial_u$ and so on. Hence, a solution of the nonlinear differential equation

$$\omega_{uu} - \omega_{vv} = \sin\omega \tag{2.1.1}$$

provides a pseudosphere [2]. In the coordinates

$$\xi = \tfrac{1}{2}(u+v), \quad \eta = \tfrac{1}{2}(u-v),$$

Equation (2.1.1) takes the form

$$\omega_{\xi\eta} = \sin\omega, \tag{2.1.2}$$

which is called the sine-Gordon equation. This can be derived as the Eular-Lagrange equation from the Lagrangian density

$$L = \tfrac{1}{2}\omega_\xi\omega_\eta - \cos\omega.$$

The sine-Gordon equation emerges also in nonlinear optics [4], quantum field theory [3], and condenced matter physics [8]. If we identify the coordinates (u, v) with the space-time coordinates (x, t), the Lorentz transformation

$$x \to x' = (1-c^2)^{-1/2}(x+ct),$$

$$t \to t' = (1-c^2)^{-1/2}(cx+t),$$

leaves Equation (2.1.1) invariant. In the coordinates (ξ, η), this implies Lie invariance of the sine-Gordon equation under the scale transformation of the independent variables,

$$\xi \to \xi' = \varepsilon^{-1}\xi, \quad \eta \to \eta' = \varepsilon\eta, \tag{2.1.3}$$

where nonzero real constant ε is defined by $\varepsilon^2 = (1-c) \times(1+c)^{-1}$.

An important class of two-dimensional nonlinear classical fields taking their values in Riemannian manifolds has been investigated for these several years. Let us discuss one of the simplest examples called $O(n)$ invariant sigma-models. The motion of a string of n-dimensional classical spin of unit length is described by an S^n-valued field. The Lagrangian density is defined by

$$L(x) = \tfrac{1}{2}(\partial_\mu q.\partial^\mu q) + \tfrac{1}{2}\lambda(q^2 - 1),$$

with the short hand notation: $x = (x^0, x^1)$, $q = (q_1, \ldots, q_n)$, $\partial_\mu = \partial/\partial_{x^\mu}$, $\partial^\mu = \partial/\partial_{x_\mu}$, $q^2 = (q.q)$, where λ is a Lagrange multiplier [7]. Here and in what follows we use the summation convention. It is easy to see that $L(x)$ is invariant under the internal symmetry group $O(n)$. In the coordinates $\xi = 1/2(x^0 + x^1)$, $\eta = 1/2(x^0 - x^1)$, the Eular-Lagrange equations get the form

$$\partial_\xi\partial_\eta q + (\partial_\xi q.\partial_\eta q)q = 0, \quad q^2 = 1. \tag{2.1.4}$$

The sum and difference of the energy and momentum densities are given by $1/2(\partial_\xi q)^2$ and $1/2(\partial_\eta q)^2$, respectively, hence the energy-momentum conservation is expressed as

$$((\partial_\xi q)^2)_\eta = 0, \quad ((\partial_\eta q)^2)_\xi = 0.$$

It follows that $(\partial_\xi q)^2$ and $(\partial_\eta q)^2$ be of the form: $(\partial_\xi q)^2 = h^2(\xi)$, $(\partial_\eta q)^2 = k^2(\eta)$. Incidentally, the field equations (2.1.4) are invariant under the scale transformation

$$d\xi \to |H(\xi)|d\xi, \quad d\eta \to |K(\eta)|d\eta.$$

If we choose $|H(\xi)| = |h(\xi)|$ and $|K(\eta)| = |k(\eta)|$, we may normalize $\partial_\xi q$ and $\partial_\eta q$ to be

$$(\partial_\xi q)^2 = 1, \quad (\partial_\eta q)^2 = 1,$$

$$-1 \leq (\partial_\xi q.\partial_\eta q) \leq 1. \qquad (2.1.5)$$

These break the scale invariance of Equations (2.1.4). For $n = 3$, the unit vectors q, $\partial_\xi q$, and $\partial_\eta q$ spanning the space $\mathbb{R}^3$ satisfy $(q.\partial_\xi q) = 0$, $(q.\partial_\eta q) = 0$, therefore the frame defines the nontrivial angle ω through

$$(\partial_\xi q.\partial_\eta q) = -\cos\omega. \qquad (2.1.6)$$

From Equation (2.1.4) to (2.1.6) it follows, after a caluculation, that the ω satisfies the sine-Gordon equation

$$\omega_{\xi\eta} = \sin\omega.$$

.

Noting that $\mathbb{R}^3$, the q-space, is viewed as a tangent space to $SU(2)$ at the identity, we define an $SU(2)$ matrix g by

$$g = \begin{pmatrix} iq_1 & iq_2 - q_3 \\ iq_2 + q_3 & -iq_1 \end{pmatrix} = iq_k\sigma^k, \tag{2.1.7}$$

where σ^k denote the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The $O(3)$ nonlinear sigma-models are then put into the form

$$\partial_\xi(\partial_\eta g\cdot g^{-1}) + \partial_\eta(\partial_\xi g\cdot g^{-1}) = 0. \tag{2.1.8}$$

These are called the $SU(2)$ chiral field equations and can be derived from the Lagrangian density

$$L(\xi,\ \eta) = \frac{1}{2}\,\mathrm{Tr}(\partial_\xi g\cdot\partial_\eta g^{-1}), \quad g \in SU(2).$$

Let G be a matrix Lie group and g its Lie algebra. For a G-valued function $g(x)$, the Lagrangian density

$$L(x) = \frac{1}{2}\,\mathrm{Tr}(\partial_\mu g\cdot\partial^\mu g^{-1}), \quad g \in G \tag{2.1.9}$$

gives rise to the field equations

$$\partial^\mu A_\mu = 0 \quad \text{with} \quad A_\mu = g^{-1}\partial_\mu g \in g, \tag{2.1.10}$$

which are referred to as the G chiral field equations. The g-valued matrices A_μ satisfy a zero-curvature condition

$$\varepsilon^{\mu\nu}(\partial_\mu A_\nu + A_\mu A_\nu) = 0, \tag{2.1.11}$$

where $\varepsilon^{\mu\nu}$ are skew-symmetric tensor with $\varepsilon^{01} = 1$. The chiral field equations (2.1.10) are closely related with a topological property of the manifold G. For example, the number of instanton-like solutions is characterized by the second homotopy group of G [6].

2.2. Bäcklund transformation and Riemann-Hilbert problem

As was shown in the previous section, the sine-Gordon equation (2.1.2) is invariant under the one-parameter group of transformation (2.1.3), hence if $\omega(\xi, \eta)$ satisfies the sine-Gordon equation, so does $\omega(\varepsilon^{-1}\xi, \varepsilon\eta)$. This solution is said to be obtained by a Lie transformation L_ε.

Bianchi [2] proposed another useful transformation from a given solution $\omega(\xi, \eta)$ into a new solution $\omega'(\xi, \eta)$, which we mean by B. Associating the Lie transformation with Bianchi's transformation, Bäcklund presented a transformation

$B_\varepsilon = L_\varepsilon^{-1} \circ B \circ L_\varepsilon$ as follows.

Theorem 2.2.1, (Bäcklund [2]). *Let* $\omega(\xi, \eta)$ *be a solution of the sine-Gordon equation. Then* $\omega'(\xi, \eta)$ *defined by*

$$\omega'_\eta - \omega_\xi = 2\varepsilon\sin\{\tfrac{1}{2}(\omega' + \omega)\},$$

$$\omega'_\xi + \omega_\xi = 2\varepsilon^{-1}\sin\{\tfrac{1}{2}(\omega' - \omega)\} \qquad (2.2.1)$$

is another solution of the sine-Gordon equation.

The proof is carried out by calculation of the integrability condition for $\omega'_{\xi\eta} = \omega'_{\eta\xi}$. The system (2.2.1) is referred to as *the Bäcklund transformation* of the sine-Gordon equation. It is to be remarked that for $\varepsilon = 1$ the Bäcklund transformation B_ε reduces to Bianchi's transformation B. We mean (2.2.1) by $B_\varepsilon : \omega \to \omega'$. Suppose we have a set of solutions ω_k, $0 \le k \le 4$, determined by $B_{\varepsilon_1}: \omega_0 \to \omega_1$, $B_{\varepsilon_2}: \omega_1 \to \omega_2$, $B_{\varepsilon_2}: \omega_0 \to \omega_3$, and $B_{\varepsilon_1}: \omega_3 \to \omega_4$, where $\varepsilon_1 \neq \varepsilon_2$. It is a matter of calculation to see that $\omega_2 = \omega_4$ up to integration constants. This fact is known as *the theorem of permutability* and demonstrates effectiveness of the Bäcklund transformation. The trivial solution $\omega = 0$ is a basis for constructing multi-soliton solutions. Bianchi [2] also derived the equation

$$\tan\{\tfrac{1}{4}(\omega_2 - \omega_0)\} = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 - \varepsilon_2} \tan\{\tfrac{1}{4}(\omega_1 - \omega_3)\}. \qquad (2.2.2)$$

Solving this yields multi-soliton solutions.

Next we discuss the relationship between the Bäcklund transformation and the inverse scattering formula. Set $\Gamma(\xi, \eta) = \tan\{1/4(\omega' + \omega)\}$. Then from (2.2.1), we have the Riccati equations

$$\Gamma_\eta = \frac{1}{2}\,\omega_\eta(1 + \Gamma^2) + \varepsilon\Gamma,$$

$$\Gamma_\xi = -\,(2\varepsilon)^{-1}\sin\omega\cdot(1 - \Gamma^2) + \varepsilon^{-1}\cos\omega\cdot\Gamma.$$

Introducing dependent variables y_1 and y_2 through $\Gamma(\xi, \eta) = y_2 y_1^{-1}$, we can bring out the system of linear differential equations $L_1\Phi = 0$ and $L_2\Phi = 0$ with

$$L_1 = \begin{pmatrix} \partial_\eta & 0 \\ 0 & \partial_\eta \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -\varepsilon & -\omega_\eta \\ \omega_\eta & \varepsilon \end{pmatrix},$$

$$L_2 = \begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\xi \end{pmatrix} + \frac{1}{2\varepsilon}\begin{pmatrix} \cos\omega & \sin\omega \\ \sin\omega & -\cos\omega \end{pmatrix},$$

where $\Phi = {}^t(y_1, y_2)$. This linearization procedure of course is not unique. The compatibility condition

$$[L_1, L_2] = 0 \qquad (2.2.3)$$

for the linear operators L_1 and L_2 is equivalent to the

the sine-Gordon equation. Then this system is called the *inverse scattering formula* for the sine-Gordon equation. Ablowitz-Kaup-Newell-Segur [1] solved its inverse scattering problem.

It is of practical use to write the inverse scattering formula in the form

$$\partial_\eta Y = U(\omega)Y, \quad \partial_\xi Y = V(\omega)Y \tag{2.2.4}$$

$$\text{with} \quad Y = Y(\xi, \eta; \zeta) = \begin{pmatrix} y_1 & -y_2^* \\ y_2 & y_1^* \end{pmatrix},$$

$$U(\omega) = \frac{1}{2}\begin{pmatrix} -i\zeta & -\omega_\eta \\ \omega_\eta & i\zeta \end{pmatrix}, \quad V(\omega) = \frac{i}{2\zeta}\begin{pmatrix} \cos\omega & \sin\omega \\ \sin\omega & -\cos\omega \end{pmatrix},$$

where we use $\zeta \in \mathbb{R}$, $\zeta \neq 0$, in the place of ε, and $*$ denotes the complex conjugate. Using this notation, we show the following theorem about the Bäcklund transformation.

Theorem 2.2.2. *Let a unitary matrix* G *and its inverse* G^{-1} *be*

$$G = \frac{1}{\zeta + i\varepsilon}\begin{pmatrix} \zeta + i\varepsilon\cos\Omega & i\varepsilon\sin\Omega \\ i\varepsilon\sin\Omega & \zeta - i\varepsilon\cos\Omega \end{pmatrix},$$

$$G^{-1} = \frac{1}{\zeta - i\varepsilon}\begin{pmatrix} \zeta - i\varepsilon\cos\Omega & -i\varepsilon\sin\Omega \\ -i\varepsilon\sin\Omega & \zeta + i\varepsilon\cos\Omega \end{pmatrix},$$

respectively, where $\Omega = 1/2(\omega' + \omega)$.

Suppose functions $\omega = \omega(\xi, \eta)$ *and* $\omega' = \omega'(\xi, \eta)$ *are* *related by the Bäcklund transformation* (2.2.1). *Then a matrix* Y' *defined by*

$$Y' = GY \tag{2.2.5}$$

satisfies

$$\partial_\eta Y' = U(\omega')Y', \quad \partial_\xi Y' = V(\omega')Y' \tag{2.2.6}$$

Proof. Substituting $Y = G^{-1}Y'$ into the system (2.2.4), we obtain

$$\partial_\eta Y' = (GU(\omega)G^{-1} + \partial_\eta G\cdot G^{-1})Y',$$

$$\partial_\xi Y' = (GV(\omega)G^{-1} + \partial_\xi G\cdot G^{-1})Y'.$$

Each coefficient matrix can be written out to be

$$GU(\omega)G^{-1} + \partial_\eta G\cdot G^{-1}$$

$$= U(\omega') - \frac{\zeta(\omega'_\eta - \omega_\eta - 2\varepsilon\sin\Omega)}{2(\zeta^2 + \varepsilon^2)}\begin{pmatrix} i\varepsilon\sin\Omega & -\zeta - i\varepsilon\cos\Omega \\ \zeta - i\varepsilon\cos\Omega & -i\varepsilon\sin\Omega \end{pmatrix},$$

$$GV(\omega)G^{-1} + \partial_\xi G\cdot G^{-1}$$

$$= V(\omega') - \frac{\zeta\varepsilon(\omega'_\xi + \omega_\xi + 2\varepsilon^{-1}\sin\Omega)}{2(\zeta^2 + \varepsilon^2)}\begin{pmatrix} -i\sin\Omega & i\varepsilon + \zeta\cos\Omega \\ -i\varepsilon + \zeta\cos\Omega & i\sin\Omega \end{pmatrix}.$$

The Bäcklund transformation then yields the system (2.2.6).□

The solution matrix $Y'(\xi, \eta; \zeta)$ gives a new solution $\omega'(\xi, \eta)$ of the sine-Gordon equation, providing that the

Bäcklund transformation holds. Let us write G as

$$G = 1 - \frac{2i\varepsilon}{\zeta + i\varepsilon} P \tag{2.2.7}$$

$$\text{with} \quad P = \frac{1}{2}\begin{pmatrix} 1 - \cos\Omega & -\sin\Omega \\ -\sin\Omega & 1 + \cos\Omega \end{pmatrix}.$$

We note that $\det P = 0$ and $P^2 = P$. The matrix P is a projection operator.

Recently Zakharov and his coworkers [10, 11] have developed a method for generating exact solutions in terms of the Riemann-Hilbert problem of the inverse scattering formulae. By using the method, multi-soliton solutions can be constructed without solving integral equations emerging in the usual inverse scattering method. Their one-soliton transformation $\hat{G}$ is also related to a projection operator $\hat{P}$ in a way such that

$$\hat{Y} = \hat{G}Y, \quad \hat{G} = 1 - \frac{2i\varepsilon}{\zeta + i\varepsilon} \hat{P}, \tag{2.2.8}$$

where $\hat{Y}$ is a matrix satisfing $\partial_\eta \hat{Y} = U(\hat{\omega})\hat{Y}$ and $\partial_\xi \hat{Y} = V(\hat{\omega})\hat{Y}$. In the remainder of this section we shall derive the projection operator $\hat{P}$ after Zakharov-Mikhailov [10]. Note that the matrices $U(\hat{\omega})$ and $V(\hat{\omega})$ are given by

$$U(\hat{\omega}) = \hat{G}(U(\omega) - \partial_\eta)\hat{G}^{-1},$$

$$V(\hat{\omega}) = \hat{G}(V(\omega) - \partial_\xi)\hat{G}^{-1}.$$

Since the residues of $U(\hat{\omega})$ and $V(\hat{\omega})$ at the poles $\zeta = \pm i\varepsilon$ are required to vanish, it follows from the above equation that

$$(1 - \hat{P})(U(\omega;\ \zeta = i\varepsilon) - \partial_\eta)\hat{P} = 0,$$

$$\hat{P}(U(\omega;\ \zeta = -i\varepsilon) - \partial_\eta)(1 - \hat{P}) = 0,$$

$$(1 - \hat{P})(V(\omega;\ \zeta = i\varepsilon) - \partial_\xi)\hat{P} = 0,$$

$$\hat{P}(V(\omega;\ \zeta = -i\varepsilon) - \partial_\xi)(1 - \hat{P}) = 0. \tag{2.2.9}$$

We observe that the conditions (2.2.9) are satisfied if the operator $\hat{P}$ is characterized by the equations

$$(1 - \hat{P})\vec{m} = 0, \quad \hat{P}\vec{n} = 0, \tag{2.2.10}$$

where $\vec{m} = Y(\zeta = i\varepsilon)e_1$, $\vec{n} = Y(\zeta = -i\varepsilon)e_2$, $e_1 = {}^t(1,\ 0)$, and $e_2 = {}^t(0,\ 1)$. From $y_j(\zeta = \pm i\varepsilon) = y_j^*(\zeta = \mp i\varepsilon)$, we can show that $(\vec{m},\ \vec{n}) = 0$. The proof is performed by operating on $\vec{m}$ and $\vec{n}$ with the right-hand sides of (2.2.9) and using the inverse scattering formula (2.2.6) with (2.2.10). It follows from (2.2.10) that

$$\hat{p}_{jk} = \frac{m_j m_k^*}{|m_1|^2 + |m_2|^2}, \tag{2.2.11}$$

where $\hat{P} = (\hat{p}_{jk})_{j,k=1,2}$ and $\vec{m} = {}^t(m_1,\ m_2)$.

The projection operator defined by (2.2.11) produces (N+1)-soliton solution to the sine-Gordon equation from N-soliton solution. We notice here that the P given by (2.2.7) satisfies Equations (2.2.10). Thus we infer that the operator P is essentially equivalent to the operator $\hat{P}$. The following algebraic relations of ω' and ω is as immediate consequence of (2.2.7) and (2.2.10).

Proposition 2.2.3. *Let* ω *and* Y *satisfy* (2.2.4). *Then a new solution* ω' *of the sine-Gordon equation is given by*

$$\sin \omega' = \{(p_{12}+p_{21})^2 - (p_{11}-p_{22})^2\}\sin\omega$$

$$+ 2(p_{12}+p_{21})(p_{11}-p_{22})\cos\omega\ ,$$

$$\cos \omega' = 2(p_{12}+p_{21})(p_{11}-p_{22})\sin\omega$$

$$- \{(p_{12}+p_{21})^2 - (p_{11}-p_{22})^2\}\cos\omega\ .$$

Proof. When $\zeta = i\varepsilon$, $U(\omega)$ and $V(\omega)$ are real. Then the ratio of $y_1(\zeta = \pm i\varepsilon)$ and $y_2(\zeta = \pm i\varepsilon)$ becomes real. From (2.2.8) and (2.2.9), we have

$$(1+\cos\Omega)y_1(\zeta = i\varepsilon) + \sin\Omega\cdot y_2(\zeta = i\varepsilon) = 0,$$

$$\sin\Omega\cdot y_1(\zeta = i\varepsilon) + (1-\cos\Omega)y_2(\zeta = i\varepsilon) = 0.$$

Thus $\cos\Omega$ and $\sin\Omega$ are expressed in terms of y_1 and y_2. Further calculation with (2.2.10) shows that $\sin\Omega = -p_{12}-p_{21}$, $\cos\Omega = -p_{11}+p_{22}$. This proves the proposition. □

In summary, the well-known Bäcklund transformation (2.2.1) takes the form of gauge transformation (2.2.5) for the inverse scattering formula (2.2.4). Under the relations (2.2.10), the gauge transformation (2.2.5) is equivalent to the one-soliton transformation (2.2.8) in the Riemann-Hilbert problem. We shall call a transformation like (2.2.8) with $\hat{P}$ in (2.2.11) the Bäcklund transformation in subsequent chapters.

2.3. Inverse scattering formula for chiral field equations

The inverse scattering formula for a nonlinear equation is a pair of linear differential equations with a parameter whose compatibility condition is the original nonlinear equation. A systematic method for finding such linear equations was proposed by Wahlquist and Estabrook [9]. We here give a brief review of their idea called *the method of prolongation.*

Let $I(\alpha)$ be a differential ideal, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ are exterior differential forms of rank 2 (2-forms) such that the exterior differential equations $\alpha = 0$ derive a solution manifold of a system of nonlinear differential

. .

equations. We choose local coordinates (u, x) of the space on which $I(\alpha)$ is defined so that the solution manifold may be described by $u = (u_1, \ldots, u_m)$ and $x = (x^0, x^1)$ as dependent and independent variables, respectively. If the exterior differentiation of a 1-form

$$\theta = U(u, x)dx^0 + V(u, x)dx^1 \tag{2.3.1}$$

belongs to the ideal $I(\alpha)$, that is, $d\theta \in I(\alpha)$, then θ yields along the solution a conservation laws

$$\partial_1 U - \partial_0 V = 0 \tag{2.3.2}$$

for the nonlinear differential equations. Here the form θ can be extended to be matrix valued. Wahlquist and Estabrook studied a *prolongation* of (2.3.1) in an extended space of variables. Consider the following form Θ in the extended space of variables x, u, and Y,

$$\Theta = -dY + U(u, x)Y\,dx^0 + V(u, x)Y\,dx^1 , \tag{2.3.3}$$

where Y, U and V are $n \times n$ matrices. Requiring that $d\Theta \in I(\alpha, \Theta)$, we have an overdetermined system of partial differential equations for U and V. We have assumed here that Y is a matrix function of x. If there is a non-trivial solution for the equations obtained, we derive from (2.3.3) an inverse scattering formula

$$\partial_0 Y = U(u, x; \zeta)\, Y, \qquad \partial_1 Y = V(u, x; \zeta)\, Y. \tag{2.3.4}$$

Here we have taken a parameter ζ in U and V which comes from the representation of a Lie algebra U and V take their values in.

In this section, we generalize the method of prolongation to obtain an inverse scattering formula for matrix valued nonlinear differential equations. For example, consider the chiral field equations (2.1.10). By the analytic continuation of independent variables into the complex space $(y, z) \in \mathbb{C}^2$ and by defining new variables; $A = \partial_y g \cdot g^{-1}$, $B = \partial_z g \cdot g^{-1}$, we convert the G chiral field equations (2.1.10) to the first order equations

$$\partial_z A + \partial_y B = 0. \tag{2.3.5}$$

The real section $(y, z) \in \mathbb{R}^2$ and the complex conjugate section $y = z^*$ correspond to the Minkowski space and the Euclidean space, respectively. From the definition of A and B, the zero curvature condition (2.1.11),

$$\partial_z A - \partial_y B + [A, B] = 0, \tag{2.3.6}$$

should hold. Equations (2.3.5) and (2.3.6) take the place of the G chiral field equations. We introduce a set of 2-forms by

$$\alpha_1 = dA \wedge dy - dB \wedge dz,$$

$$\alpha_2 = dA \wedge dy + dB \wedge dz - [A, B]\, dy \wedge dz \qquad (2.3.7)$$

on the space of variables A, B, y and z. Note that A and B are in $\mathcal{g}$, the Lie algebra of G. The equivalence between the exterior differential equations, $\alpha_1 = 0$, $\alpha_2 = 0$, and the field equations (2.3.5) and (2.3.6) can be guaranteed by $d\alpha_k \in I(\alpha_1, \alpha_2)$, $k = 1, 2$. We introduce an $n \times n$ matrix variable Y in addition to the above variables. In the prolonged space of variables A, B, y, z and Y, we search for Pfaffian

$$\Theta = -dY + U(A, B)\, Y\, dy + V(A, B)\, Y\, dz$$

whose differential $d\Theta$ is in the prolonged ideal $I(\alpha_1, \alpha_2, \Theta)$ under the condition that Y is a matrix function of y and z. We now show a sufficient condition for Θ.

Proposition 2.3.1. *If the Phaffian* Θ *takes the form*

$$\Theta = -dY + \frac{\zeta}{1+\zeta} AY\, dy - \frac{\zeta}{1-\zeta} BY\, dz \qquad (2.3.8)$$

$\zeta \in \mathbb{C}$, $\zeta \neq \pm 1$, *then* $d\Theta \in I(\alpha_1, \alpha_2, \Theta)$.

Proof. Substituting (2.3.7) and (2.3.8) into the exterior differentiation of Θ, we find

$$d\Theta = \frac{\zeta}{1-\zeta^2}\alpha_1 Y - \frac{\zeta^2}{1-\zeta^2}\alpha_2 Y - (\frac{\zeta}{1+\zeta}A - \frac{\zeta}{1-\zeta}B)\Theta.$$

This ends the proof. □

Like (2.3.4) this proposition gives an inverse scattering formula for the G chiral field equations

$$\partial_y Y = \frac{\zeta}{1+\zeta}AY, \quad \partial_z Y = -\frac{\zeta}{1-\zeta}BY. \tag{2.3.9}$$

Remark 2.3.2. Since $\partial_y Y \cdot Y^{-1} = \partial_y g \cdot g^{-1}$ and $\partial_z Y \cdot Y^{-1} = \partial_z g \cdot g^{-1}$ at the limit $\zeta \to \infty$, the solution matrix $Y = Y(y, z; \zeta)$ yields a solution of the chiral field equations by setting $Y(\infty) = g$.

2.4. Nonlocal conservation laws

The sine-Gordon equation (2.1.2) possesses an infinite number of local conservation laws. The adjective *local* denotes each conserved current to be expressed by the field ω and its higher derivatives. The Bäcklund transformation (2.2.1) gives rise to

$$\varepsilon(\mathrm{coc}\{\tfrac{1}{2}(\omega' + \omega)\})_\xi - \varepsilon^{-1}(\cos\{\tfrac{1}{2}(\omega' - \omega)\})_\eta = 0. \tag{2.4.1}$$

Expanding $\omega' = \omega'(\xi, \eta\ ; \zeta)$ in the formal power series around

$\varepsilon = 0$ and collecting terms involving the same power of ε, we have an infinite number of local conservation laws.

Next we discuss for the chiral field equations the *nonlocal* conservation laws which are described in terms of integrals of field variables. We write the inverse scattering formula (2.3.9) in the form

$$(1+\zeta)\partial_y Y = \zeta AY, \quad (1-\zeta)\partial_z Y = -\zeta BY. \tag{2.4.2}$$

Let $Y^{(n)}$ be the coefficients of the expansion of $Y = Y(\zeta)$

$$Y(\zeta) = \sum_{n=0}^{\infty} Y^{(n)} \zeta^n. \tag{2.4.3}$$

Inserting (2.4.3) into (2.4.2), we obtain

$$\begin{aligned} \partial_y Y^{(n)} &= -\partial_y Y^{(n-1)} + A\, Y^{(n-1)}, \quad \partial_y Y^{(0)} = 0, \\ \partial_z Y^{(n)} &= -\partial_z Y^{(n-1)} - B\, Y^{(n-1)}, \quad \partial_z Y^{(0)} = 0, \end{aligned} \tag{2.4.4}$$

$n = 1, 2, \cdots$. We can assume $Y^{(0)} = 1$ without loss of generality. From (2.4.4) we get an infinite number of nonlocal conservation laws

$$\begin{aligned} &\partial_z(\partial_y Y^{(n-1)} - AY^{(n-1)}) \\ &+ \partial_y(\partial_z Y^{(n-1)} - BY^{(n-1)}) = 0. \end{aligned} \tag{2.4.5}$$

These are essentially equivalent to the conservation laws of Lüsher and Pohlmeyer [5].

CHAPTER III

INVERSE SCATTERING FORMULA FOR THE EINSTEIN EQUATIONS

The inverse scattering method to solve the Korteweg-de Vries (K-dV) equation [4] was first developed by use of a Schrödinger equation, $y_{xx} - uy = \lambda y$, whose potential u is a solution of the K-dV equation. This technique was soon expressed in a general form by Lax [6]. He serached for inverse scattering formulae for other nonlinear equations, such as the Schrödinger equation associated with the K-dV equation. The resulting inverse scattering formula, $Ly = \lambda y$, $y_t = Ay$, is frequently referred to as the Lax pair. The hermitian operator A plays a fundamental role in his theory. Later in 1974, Zakharov and Shabat [10] discussed a Dirac-type inverse scattering formula, $\partial_x Y = UY$, $\partial_t Y = VY$, in the 2×2 matrix form. This is called the Zakharov-Shabat system. A point to be made is that the matrices U and V do need not to be hermitian. Once we can find the Lax pair or the Zakharov-Shabat system, it is possible to integrate the original nonlinear equation by the method of inverse scattering.

In this chapter, we show the existence of an inverse scattering formula for Einstein's gravitational field equations in general relativity. The stationary axially symmetric vacuum Einstein equations are reviewed in Section 3.1, which contain an independent variable explicitly. Namely, the field equations are nonautonomous. No inverse scattering formulae for such equations have been known yet.

Recently, Maison [7] has found a Zakharov-Shabat system for the above Einstein equations after the analogy of the chiral field equations. However, it is hard to construct exact solutions of the Einstein equations by solving this inverse scattering formula. Because the compatibility condition of his system is a generalized sine-Gordon equation, rather difficult to deal with. Belinsky and Zakharov [1] have proposed another system which has a derivative with respect to a complex parameter. The Ernst form [2] of the stationary axially symmetric Einstein equations is useful in finding new solutions. In Section 3.2, we introduce a system of first-order differential equations associated to the Ernst equation. Applying the method of prolongation to the system, we obtain an inverse scattering formula for the Ernst equation which is different from those in [1, 7]. Finally in Section 3.3, a geometrical meaning of the inverse scattering formula is studied from the viewpoint of the principal fiber bundle. Local conservation laws are also derived via a Riccati equation.

3.1. Stationary axially symmetric vacuum Einstein's gravitational field equations

The metric of a stationary axially symmetric space-time can be written as

$$-ds^2 = e^{\Gamma}(d\rho^2 + dz^2) + q_{ij}dx^i dx^j, \qquad (3.1.1)$$

where Γ and q_{ij}, $(i, j = 1, 2)$, are real functions of ρ and z, and $Q = (q_{ij})$ is symmetric. We use the notation $(x^0, x^1, x^2, x^3) = (\rho, t, \phi, z)$, the cylindrical coordinates. Let us impose the supplementary condition

$$\det Q = -\rho^2.$$

Substituting the metric (3.1.1) into Einstein's gravirational field equations in vacuuo, $R_{\mu\nu} = 0$, where $0 \le \mu, \nu \le 3$, we have a system of differential equations

$$\begin{aligned} \partial_\rho \Gamma &= -\rho^{-1} + \tfrac{1}{4}\rho \operatorname{Tr}\{(\partial_\rho Q\cdot Q^{-1})^2 - (\partial_z Q\cdot Q^{-1})^2\}, \\ \partial_z \Gamma &= \tfrac{1}{2}\rho \operatorname{Tr}(\partial_\rho Q\cdot Q^{-1}\partial_z Q\cdot Q^{-1}), \end{aligned} \qquad (3.1.2)$$

$$\partial_\rho(\rho\partial_\rho Q\cdot Q^{-1}) + \partial_z(\rho\partial_z Q\cdot Q^{-1}) = 0. \qquad (3.1.3)$$

From (3.1.2) we observe that the function Γ is determined by quadrature once q_{ij} are known. Thus Γ may be ignored in our discussion.

We concentrate on a particular factrization (f, ω)

due to Papapetrou [8],

$$Q = \begin{pmatrix} f & f\omega \\ f\omega & f\omega^2 - \rho^2 f^{-1} \end{pmatrix} . \tag{3.1.4}$$

From the field equations (3.1.3) with (3.1.4), we have

$$\begin{aligned} & f(\partial_\rho{}^2 + \rho^{-1}\partial_\rho + \partial_z{}^2) f - (\partial_\rho f)^2 - (\partial_z f)^2 \\ & \quad + (\rho^{-1} f^2 \partial_\rho \omega)^2 + (\rho^{-1} f^2 \partial_z \omega)^2 = 0, \end{aligned} \tag{3.1.5}$$

$$\partial_\rho(\rho^{-1} f^2 \partial_\rho \omega) + \partial_z(\rho^{-1} f^2 \partial_z \omega) = 0. \tag{3.1.6}$$

Equation (3.1.6) implies that there exists a function $\psi(\rho, z)$ satisfying

$$\partial_\rho \psi = \rho^{-1} f^2 \partial_z , \quad \partial_z \psi = - \rho^{-1} f^2 \partial_\rho \omega. \tag{3.1.7}$$

The function ψ is called a twist potential. Eliminating ω in (3.1.5) and (3.1.7), we obtain

$$\begin{aligned} & f(\partial_\rho{}^2 + \rho^{-1}\partial_\rho + \partial_z{}^2) f - (\partial_\rho f)^2 - (\partial_z f)^2 \\ & \quad + (\partial_\rho \psi)^2 + (\partial_z \psi)^2 = 0. \end{aligned} \tag{3.1.8}$$

From (3.1.7) the integrability condition for ω results in

$$\partial_\rho(\rho f^{-2} \partial_\rho \psi) + \partial_z(\rho f^{-2} \partial_z \psi) = 0. \tag{3.1.9}$$

Equations (3.1.8) and (3.1.9) are the field equations to be investigated. Ernst [2] introduced a complex potential

$$E = f + i\psi, \tag{3.1.10}$$

in order to cast Equations (3.1.8) and (3.1.9) into a single equation

$$f(\partial_\rho{}^2 + \rho^{-1}\partial_\rho + \partial_z{}^2)\, E - (\partial_\rho E)^2 - (\partial_z E)^2 = 0. \tag{3.1.11}$$

The real and imaginary parts of (3.1.11) are (3.1.8) and (3.1.9), respectively. Equation (3.1.11) is referred to as the Ernst equation. It is worthwhile to express the Ernst equation as

$$(\mathrm{Re}\, E)\, \Delta E - \nabla E \cdot \nabla E = 0, \tag{3.1.12}$$

which does not refer to a particular coordinate system. The Ernst equation is derived from the Lagrangian density

$$L = (\mathrm{Re}\, E)^{-2}\, \nabla E \cdot \nabla E^*, \tag{3.1.13}$$

where $*$ indicates the complex conjugate. Ernst defined an alternative complex potential Ξ by a mapping $\Xi = (1 + E)(1 - E)^{-1}$. Then Ξ satisfies

$$(\Xi\Xi^* - 1)\Delta\Xi - 2\Xi^*\nabla\Xi\cdot\nabla\Xi.$$

This form of the Ernst equation is of wide application. For example, important classes of exact solutions such as the Schwarzchild-Weyl solutions and the Kerr-Tomimatsu-Sato solutions have been found out in the prolate spheroidal coordinates.

3.2. Inverse scattering formula for the Ernst equation

Let us set (ξ, η) as complex conjugate coordinates,

$$\xi = \frac{1}{2}(\rho + iz), \quad \eta = \frac{1}{2}(\rho - iz). \tag{3.2.1}$$

Then the Ernst equation (3.1.11), which we are going to study, reads

$$2f\,\partial_\xi\partial_\eta E + \rho^{-1} f(\partial_\xi E + \partial_\eta E) - 2\partial_\xi E\partial_\eta E = 0. \tag{3.2.2}$$

We define new dependent variables t, u, v and w by

$$\begin{aligned} t &= f^{-1}\partial_\xi E - \rho^{-1}, \quad u = f^{-1}\partial_\eta E - \rho^{-1}, \\ v &= f^{-1}\partial_\xi E^* - \rho^{-1}, \quad w = f^{-1}\partial_\eta E^* - \rho^{-1}. \end{aligned} \tag{3.2.3}$$

The Ernst equation can then be expressed as follows:

$$\begin{aligned} E_1 &= 2\partial_\eta t - tu + tw + \rho^{-1}(t + w) = 0, \\ E_2 &= 2\partial_\xi u - tu + uv + \rho^{-1}(u + v) = 0, \\ E_1^* &= 2\partial_\xi w - vw + tw + \rho^{-1}(t + w) = 0, \\ E_2^* &= 2\partial_\eta v - vw + uv + \rho^{-1}(u + v) = 0. \end{aligned} \tag{3.2.4}$$

We call the set of these first-order differential equations a reduced form of the Ernst equation.

Let us introduce a set of 2-forms which vanish for the solutions of the reduced form,

$$\alpha_1 = 2dt \wedge d\xi - \{tu - tw - \rho^{-1}(t + w)\}d\xi \wedge d\eta,$$

$$\alpha_2 = 2du \wedge d\eta + \{tu - uv - \rho^{-1}(u + v)\}d\xi \wedge d\eta,$$

$$\alpha_3 = 2dw \wedge d\eta + \{vw - tw - \rho^{-1}(t + w)\}d\xi \wedge d\eta,$$

$$\alpha_4 = 2dv \wedge d\xi - \{vw - uv - \rho^{-1}(u + v)\}d\xi \wedge d\eta.$$

Using the method of prolongation discussed in Section 2.3, we look for 1-forms

$$\Theta = - d\Phi + U\Phi\, d\xi + V\Phi\, d\eta \tag{3.2.5}$$

such that $d\Theta \in I(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \Theta)$. Here U and V are 2×2 matrix functions of t, u, v, w, ξ and η, and Φ is a two-vector. Then the 1-forms (3.2.5) give rise to a pair of linear differential operators

$$L_1 = 1\partial_\xi + U, \quad L_2 = 1\partial_\eta + V,$$

whose compatibility condition is equivalent to the reduced form. The result is as follows.

Theorem 3.2.1. *Define* L_1 *and* L_2 *as*

$$L_1 = \begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\xi \end{pmatrix} + \frac{t - v}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\zeta}{2}\begin{pmatrix} 0 & t \\ v & 0 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} \partial_\eta & 0 \\ 0 & \partial_\eta \end{pmatrix} + \frac{w - u}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2\zeta}\begin{pmatrix} 0 & w \\ u & 0 \end{pmatrix}, \qquad (3.2.6)$$

where $\zeta^2 = (\varepsilon - i\eta)\times(\varepsilon - i\xi)^{-1}$, $\varepsilon \in \mathbb{R}$. *If and only if the functions* t, u, v *and* w *satisfy the reduced form, the operators* L_1 *and* L_2 *are compatible with each other, that is,*

$$[L_1, L_2] = 0. \qquad (3.2.7)$$

Proof. Calculating the commutator, we find

$$[L_1, L_2] = \frac{1}{8}\begin{pmatrix} -E_1 - E_2 + E_1^* + E_2^* & 2(\zeta^{-1}E_1^* - \zeta E_1) \\ 2(\zeta^{-1}E_2 - \zeta E_2^*) & E_1 + E_2 - E_1^* - E_2^* \end{pmatrix}.$$

If the reduced form holds, the condition (3.2.7) is satisfied. Conversely, from the diagonal elements of $[L_1, L_2]$, the imaginary part of the Ernst equation (3.1.9) holds. Since ζ is not equal to ζ^{-1} for any finite ε, the real part (3.1.8) is also true. This completes the proof. □

In consequence of Theorem 3.2.1, by setting

$$L_1\Phi = 0, \quad L_2\Phi = 0, \quad \text{with} \quad \Phi = {}^t(y_1, y_2) \qquad (3.2.8)$$

we get a Zakharov-Shabat system for the Ernst equation. These equations can be regarded as eigenvalue equations, where y_1 and y_2 are the eigenfunctions and ε contained in L_1 and L_2 is the eigenvalue. Recently, Harrison [5] conjectures the existence of such equations as (3.2.8).

3.3. Connection in $SL(2, \mathbb{R})$ bundle

In this section, we consider a geometrical interpretation of the inverse scattering formula (3.2.8) using the terminology of exterior differential forms[3]. We first associate a connection form with (3.2.8) and discuss the corresponding curvature form. Second, we propose a method for generating an infinite number of conservation laws of the Ernst equations.

We write the inverse scattering formula (3.2.8) as

$$d\Phi = M\Phi, \quad \text{with} \quad M = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & -\theta^1 \end{pmatrix}, \tag{3.3.1}$$

where

$$\theta^1 = -\frac{t-v}{4}\, d\xi - \frac{w-u}{4}\, d\eta,$$

$$\theta^2 = -\frac{t\zeta}{2}\, d\xi - \frac{w\zeta^{-1}}{2}\, d\eta,$$

$$\theta^3 = -\frac{v\zeta}{2}\,d\xi - \frac{u\zeta^{-1}}{2}\,d\eta. \tag{3.3.2}$$

We note here that the trace of M is zero.

In what follows we think of ξ and η as real variables, and define an $sl(2, \mathbb{R})$-valued 1-form ω to be

$$\omega = \sum_{k=1}^{3} \theta^k \otimes e_k, \tag{3.3.3}$$

where the symbol $\otimes$ denotes the tensor product, and e_k, $k = 1, 2, 3$, form a base of $sl(2, \mathbb{R})$,

$$e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

This 1-form ω may define a connection in the trivial principal $SL(2, \mathbb{R})$ bundle over $\mathbb{R}^2$. Then the following theorem is obtained.

Theorem 3.3.1. *The reduced form* (3.2.4) *of the Ernst equation is equivalent to*

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{3.3.4}$$

Proof. The 2-form Ω can be rewritten as

$$\Omega = \sum_{k=1}^{3} d\theta^k \otimes e_k + \frac{1}{2} \sum_{j,k=1}^{3} \theta^j \wedge \theta^k \otimes [e_j, e_k].$$

It is an easy matter to get the followings.

$$[e_1, e_2] = -2e_2, \quad [e_2, e_3] = -e_1,$$

$$[e_3, e_1] = -2e_3.$$

Using these relations together with (3.3.2), we calculate Ω explicitely. The result is

$$\Omega = -\frac{1}{8}(E_1 + E_2 - E_1^* - E_2^*)d\xi \wedge d\eta \otimes e_1$$
$$-\frac{1}{4}(\zeta^{-1}E_1^* - \zeta E_1)d\xi \wedge d\eta \otimes e_2$$
$$-\frac{1}{4}(\zeta^{-1}E_2 - \zeta E_2^*)d\xi \wedge d\eta \otimes e_3.$$

This proves the theorem. □

Remark 3.3.2. Equation (3.3.4) means that the curvature of the connection defined by (3.3.3) vanishes.

Next, we establish a relation between the 1-forms (3.3.2) and the pseudopotential equation originally introduced by Wahlquist-Estabrook [9]. For this purpose, we define a 1-form

$$\sigma = d\phi + \phi^2\theta^2 + 2\phi\theta^1 - \theta^3, \tag{3.3.5}$$

where ϕ is a function of ξ and η. If the reduced form of the Ernst equation (3.2.4) are satisfied, this 1-form is completely integrable, that is,

$$d\sigma = -2\chi \wedge \sigma, \quad \chi = \phi\theta^2 + \theta^1. \tag{3.3.6}$$

Substituting (3.3.2) into (3.3.5), we obtain

$$\sigma = d\phi - \frac{1}{2}\{t\zeta\phi^2 + (t-v)\phi - v\zeta\}d\xi$$

$$- \frac{1}{2}\{w\zeta^{-1}\phi^2 + (w-u)\phi - u\zeta^{-1}\}d\eta, \qquad (3.3.7)$$

which is the pseudopotential equation for the reduced form of the Ernst equation. We note that the Riccati equation $\sigma = 0$ holds on the solutions of the reduced form.

By applying the exterior differentiation to χ and using (3.3.5), we have

$$d\chi = (\sigma - 2\phi\theta^1 + \theta^3)\wedge\theta^2 + d\theta^2 + d\theta^1.$$

From Theorem 3.3.1, it follows that $d\theta^1 - \theta^2\wedge\theta^3 = 0$, and $d\theta^2 - 2\theta^1\wedge\theta^3 = 0$. Then we readily find that the 1-form $\chi(\zeta)$ is closed, that is, $d\chi(\zeta) = 0$ on the solutions of the reduced form. Consequently, we have

$$\partial_\xi(2\zeta^{-1}w\phi + w - u) - \partial_\eta(2\zeta t\phi + t - v) = 0, \qquad (3.3.8)$$

where ϕ is a solution of the Riccati equation $\sigma = 0$. Expanding $\zeta = \zeta(\varepsilon)$ into a power series in ε, we have from (3.3.8) an infinite number of nontrivial local conservation laws of the Ernst equation.

CHAPTER IV

TRANSFORMATION THEORY FOR THE EINSTEIN EQUATIONS

In Section 2.2, we saw that the sine-Gordon equation can be integrated by means of the Bäcklund transformation. Use of this transformation, as well as the theorem of permutability, in obtaining multi-soliton solutions from trivial one was studied by Lamb [6]. It is possible to say that a *transformation theory* concerning solutions of the sine-Gordon equation is set up by the use of the Bäcklund transformation. As was shown by Theorem 2.2.2, the Riemann-Hilbert problem can be used for building up an alternative transformation theory.

For Einstein's gravitational field equations, a number of exact solutions have been found by many authors (see [5]). In view of the recent development of soliton physics, it can be expected that the methods developed in the soliton theory may be successfully applied to the problem of finding solutions of the Einstein equations, systematically. In fact, the properties of the Einstein equations and some of known solutions are closely related to those of the soliton equations and their solutions. Several works concerning the Einstein equations have been done along the above line of thought. We have obtained indeed an inverse scattering formula for the Ernst form of the stationary axially symmetric field equations in the previous chapter. This manifests the possibility that the Einstein equations can be solved in the framework of the soliton theory.

In this chapter, we establish two kinds of transformation theories for the Einstein equations. First, in Section 4.1, a reduced system of the stationary axially symmetric vacuum Einstein equations is defined for the logarithmic derivatives of the unknown function. An inverse scattering formula being slightly different from that in Section 3.2 is derived for the reduced system. Next, we present a Bäcklund transformation by which new solutions can be obtained in principle from a given one. In Section 4.2, exact solutions are constructed from the Minkowski metric. A generalization of the Kasner solutions is achieved through an iteration of the Bäcklund transformation. Finally in Section 4.3, we discuss the Einstein equations and associated linear differential equations from the viewpoint of the Riemann-Hilbert problem. A generating function of nonlocal symmetries gives rise to an inverse scattering formula. It is then shown that *the Riemann-Hilbert transformation* can generate new solutions. The composition law of this transformation is also proved, which proposes an alternative transformation theory for the Einstein equations.

4.1. Bäcklund transformation

Recently two types of Bäcklund transformations of the stationary axially symmetric Einstein equations have been proposed though new solutions are not obtained. Using the method of prolongation, Harrison [2] has given a Bäcklund transformation of the Ernst equation. He has mentioned that if one chooses Minkowski's flat space metric as a beginning solution, resulting solutions also turn out to be flat. On the other hand, Neugebauer [7] has found a Bäcklund transformation which includes coordinate transformation as well as dependent variable transformation. This transformation obeys permutability.

In this section, we first define new complex functions in order to introduce a reduced system for the Einstein equations. As was shown in Chapter 3, the stationary axially symmetric gravitational fields can be derived from the solutions of the Ernst equation (3.1.11). Instead of (ρ, z) used there, we take complex conjugate coordinates (ξ, η) defined by (3.2.1). Then the field equation is put into (3.2.2). We note here that (3.2.2) is bilinear with respect to f and ψ. Let us introduce complex functions A and B, which depend on the logarithmic derivatives of f as follows:

$$A = \partial_\xi(\log f) + if^{-1}\partial_\xi\psi,$$

$$B = \partial_\eta(\log f) + if^{-1}\partial_\eta\psi. \qquad (4.1.1)$$

The functions A and B are related to the functions t and u in (3.2.3) by the equations: $A = t + \rho^{-1}$, $B = u + \rho^{-1}$. We remark that A and B have the following nice property. If the metric is asymptotically flat, then the functions A and B go to zero at infinity.

As a comparison with the field equations discussed in Section 2.2, we show here that invariants of the corresponding sigma-models are expressed by A and B. Let us define a three-dimensional vector $S = {}^t(S_1, S_2, S_3)$ to be

$$S_1 = \frac{1 + f^2 + \psi^2}{2f}, \quad S_2 = \frac{1 - f^2 - \psi^2}{2f}, \quad S_3 = \frac{\psi}{f},$$

in the three-dimensional pseudo-Euclidean space with signature $(- + +)$. The vector S satisfies the constraint condition

$$(S \cdot S) = -(S_1)^2 + (S_2)^2 + (S_3)^2 = -1.$$

Using this and the Lagrangian density (3.1.13), we find that the Ernst equation (3.1.12) takes the form

$$\Delta S = (\nabla S \cdot \nabla S)S. \qquad (4.1.2)$$

These are the equations of motion for an $SO(1,2)$ invariant nonlinear sigma-models. In the coordinates (ξ, η), the invariants of (4.1.2) can be given by the algebraic relations

$$(\partial_\xi S\cdot\partial_\xi S) = AB^*, \quad (\partial_\eta S\cdot\partial_\eta S) = A^*B,$$

$$(\partial_\xi S\cdot\partial_\eta S) = \tfrac{1}{2}\{|A|^2 + |B|^2\}.$$

When the metric is asymptotically flat, the vector S goes to ${}^t(1, 0, 0)$ at infinity.

We now proceed to define first-order differential equations of A and B. The Ernst equation (3.2.2) reads

$$2\partial_\xi A + A(A^* - B) + (\xi+\eta)^{-1}(A+B) = 0,$$

$$2\partial_\eta B + B(B^* - A) + (\xi+\eta)^{-1}(A+B) = 0. \qquad (4.1.4)$$

In the next section, we look for solutions of these equations which are asymptotically zero. It should be noted that such solutions can not always give the asymptotically flat metric through (4.1.1). Equations (4.1.4) are, however, meaningful in finding a larger class of solutions of the stationary axially symmetric Einstein equations. We call Equations (4.1.4) and their complex conjugate a reduced system of the Ernst equation and write the left-hand sides of them as R_1, R_2, R_1^*, R_2^*, respectively.

The reduced system is regarded as a compatibility condition of a pair of linear differential operators. We show

Proposition 4.1.1. *Define the operators* L *and* L* *by*

$$L = \begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\xi \end{pmatrix} + \frac{A - B^*}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\zeta}{2} \begin{pmatrix} 0 & A - (\xi + \eta)^{-1} \\ B^* - (\xi + \eta)^{-1} & 0 \end{pmatrix},$$

$$L^* = \begin{pmatrix} \partial_\eta & 0 \\ 0 & \partial_\eta \end{pmatrix} + \frac{A^* - B}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2\zeta} \begin{pmatrix} 0 & A^* - (\xi + \eta)^{-1} \\ B - (\xi + \eta)^{-1} & 0 \end{pmatrix}. \qquad (4.1.5)$$

If and only if the functions A *and* B *satisfy the reduced system* (4.1.4), *the operators* L *and* L* *are compatible eith each other, that is,* $[L, L^*] = 0$.

Proof. The commutator $[L, L^*]$ is written out to be

$$\frac{1}{8} \begin{pmatrix} -R_1 - R_2 + R_1^* + R_2^* & 2(\zeta^{-1} R_1^* - \zeta R_1) \\ 2(\zeta^{-1} R_2 - \zeta R_2^*) & R_1 + R_2 - R_1^* - R_2^* \end{pmatrix}.$$

Then proposition is proved along the same line as Theorem 3.2.1. □

From Proposition 4.1.1, we obtain an inverse scattering formula

$$L\Phi = 0, \quad L^*\Phi = 0 \tag{4.1.6}$$

for the reduced system, where $\Phi = {}^t(y_1, y_2)$.

Using the inverse scattering formula (4.1.6), we present a Bäcklund transformation of the reduced system. It is assumed that a metric is given beforehand. The simplest example is the Minkowski metric. Then the functions A and B are obtained from the definition (4.1.1). If new solutions $A^{(1)}$ and $B^{(1)}$ of the reduced system are derived from A and B, finding an exact solution of the Einstein equations becomes a matter of quadrature. Along this line of thought we show the following theorem.

Theorem 4.1.2. *Define the function* $\phi(\xi, \eta)$ *by* $\phi = y_1 y_2^{-1}$. *Suppose that* ϕ *is not equal to* $-\zeta^{\pm 1}$. *If the functions* A *and* B *satisfy the reduced system* (4.1.4), *then* $A^{(1)}$ *and* $B^{(1)}$ *defined by*

$$A^{(1)} = (1+\zeta\phi)(\xi+\eta)^{-1} - (\phi+\zeta\phi^2)(\zeta+\phi)^{-1}A,$$

$$B^{(1)} = (1+\zeta\phi)\{\zeta\phi(\xi+\eta)\}^{-1} - (1+\zeta\phi)(\zeta\phi+\phi^2)^{-1}B \tag{4.1.7}$$

also satisfy the reduced system.

Proof. From Equation (4.1.6), we obtain the Riccati equations

$$\partial_\xi\phi = \tfrac{1}{2}[-\{B^* - (\xi+\eta)^{-1}\}\zeta - (A+B^*)\phi$$
$$+ \{A - (\xi+\eta)^{-1}\}\zeta\phi^2],$$

$$\partial_\eta\phi = \tfrac{1}{2}[-\{B - (\xi+\eta)^{-1}\}\zeta^{-1} - (A^*+B)\phi$$
$$+ \{A^* - (\xi+\eta)^{-1}\}\zeta^{-1}\phi^2]. \qquad (4.1.8)$$

We substitute $A^{(1)}$ and $B^{(1)}$ for A and B in the left-hand sides of (4.1.4) and denote the resulting equations by $R_1^{(1)}$ and $R_2^{(1)}$, respectively. By using the Riccati equations (4.1.8), we find that $R_1^{(1)}$ and $R_2^{(1)}$ take the form

$$R_1^{(1)} = -(\phi+\zeta\phi^2)(\zeta+\phi)^{-1}R_1,$$
$$R_2^{(1)} = -(1+\zeta\phi)(\zeta\phi+\phi^2)^{-1}R_2.$$

Since $(1+\zeta\phi)(\zeta+\phi)^{-1}\phi^{\pm 1} \neq 0$ by the assumption, $A^{(1)}$ and $B^{(1)}$ are solutions of the reduced system (4.1.4). This completes the proof. □

Remark 4.1.3. If and only if $A = B = 0$, $\phi = -\zeta^{\pm 1}$ are solutions of Equation (4.1.8). Simultaneously, the second terms of the right-hand sides of (4.1.7) vanish. Thus we may consider that the transformations (4.1.7) includes the cases $\phi = -\zeta^{\pm 1}$.

Equations (4.1.7) gives a Bäcklund transformation for the reduced system. A similar transformation which transforms flat spaces to flat spaces has been derived by Harrison [2]. As will be seen in Section 4.2, our Bäcklund transformation maps flat spaces to non-flat spaces in general.

The corresponding solution $E^{(1)} = f^{(1)} + i\,\psi^{(1)}$ of the Ernst equation (3.2.2) is obtained from $A^{(1)}$ and $B^{(1)}$ as follows.

Proposition 4.1.4. *Let* $A^{(1)}$ *and* $B^{(1)}$ *satisfy the reduced system* (4.1.4). *If there are functions* $f^{(1)}$ *and* $\psi^{(1)}$ *satisfying*

$$d\log f^{(1)} = \frac{1}{2}(A^{(1)} + B^{*(1)})d\xi + \frac{1}{2}(A^{*(1)} + B^{(1)})d\eta,$$

$$d\,\psi^{(1)} = \frac{1}{2i}(A^{(1)} - B^{*(1)})f^{(1)}d\xi - \frac{1}{2i}(A^{*(1)} - B^{(1)})f^{(1)}d\eta, \qquad (4.1.9)$$

then $f^{(1)}$ *and* $\psi^{(1)}$ *give another solution* $E^{(1)}$ *of the Ernst equation* (3.2.2).

Proof. The total differential equations (4.1.9) are derived from the definitions (4.1.1). It is easy to show the $f^{(1)}$ and $\psi^{(1)}$ satisfy Equation (3.2.2) by direct calculations. This proves the proposition. □

We have thus completed an algorithm to construct exact solutions of the stationary axially symmetric Einstein equations.

4.2. Construction of exact solutions

In this section, we discuss an application of the Bäcklund transformation (4.1.7). We use the Minkowski metric as a known solution, that is,

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2, \qquad (4.2.1)$$

in the cylindrical coordinates (ρ, t, ϕ, z). The complex potential E and the functions A and B are then written as $E = 1$, $A = B = 0$. Equations (4.1.8) now read

$$\partial_\xi \phi = \pm\{2(\xi+\eta)\}^{-1}(\varepsilon + i\eta)^{1/2}(\varepsilon - i\xi)^{-1/2}(1-\phi^2),$$

$$\partial_\eta \phi = \pm\{2(\xi+\eta)\}^{-1}(\varepsilon - i\xi)^{1/2}(\varepsilon + i\eta)^{-1/2}(1-\phi^2).$$

Solutions of the above equations are written in the form:

$$\phi = (\kappa+\delta)(\kappa-\delta)^{-1} \quad \text{and}$$

$$(-\kappa+3\delta)(\kappa-\delta)^{-1}, \qquad (4.2.2)$$

where δ is a nonnegative constant and κ is defined by

$$\kappa = \frac{|(\varepsilon - i\xi)^{1/2} \mp (\varepsilon + i\eta)^{1/2}|}{|(\varepsilon - i\xi)^{1/2} \pm (\varepsilon + i\eta)^{1/2}|}.$$

The double sign $\mp$ and $\pm$ correspond to those in Theorem 3.2.1, $\zeta = \pm(\varepsilon + i\xi)^{1/2}(\varepsilon - i\eta)^{-1/2}$, respectively.

First, we treat the case of $\phi = (\kappa + \delta)(\kappa - \delta)^{-1}$. Making use of the Bäcklund transformation (4.1.7), we have

$$A^{(1)} = (\delta - i)(\zeta^{-1} - \zeta)[\{\delta(\zeta^{-1} + 1) - i(\zeta^{-1} - 1)\}(\xi + \eta)]^{-1},$$

$$B^{(1)} = (\delta - i)(\zeta - \zeta^{-1})[\{\delta(\zeta + 1) - i(\zeta - 1)\}(\xi + \eta)]^{-1}.$$

Equation (4.1.9) in Proposition 4.1.4 is integrated to give

$$f^{(1)} = \gamma^{(1)}(\xi + \eta)|\{(\zeta - 1)^2 - \delta^2(\zeta + 1)^2\}$$
$$\times\{(\zeta + 1)(\zeta - 1)\}^{-1}|,$$

where $\gamma^{(1)}$ is a positive constant and $f^{(1)}$ contains three parameters: ε, δ and $\gamma^{(1)}$. For the case of $\phi = (-\kappa + 3\delta)(\kappa - \delta)^{-1}$, $f^{(1)}$ can be calculated similarly. For both cases, it is not easy to obtain the twist potential $\psi^{(1)}$.

Next, we shall restrict ourselves to the special case $\delta = 0$. By β_+ and β_- we mean the Bäcklund transformations with $\phi = +1$ and $\phi = -1$, respectively. We are going to generate some solutions of Equation (3.2.2) by β_+ and β_-. For the transformation β_+, we obtain

$$A^{(1)} = (\zeta + 1)(\xi + \eta)^{-1},$$

$$B^{(1)} = (\zeta^{-1} + 1)(\xi + \eta)^{-1},$$

$$f^{(1)} = \gamma^{(1)}(\xi + \eta)|(\zeta - 1)(\zeta + 1)^{-1}|, \quad \psi^{(1)} = 0,$$

from the Minkowski metric. By the transformation β_-, new solutions $A^{(2)}$ and $B^{(2)}$ of the reduced system (4.1.4) are derived from $A^{(1)}$ and $B^{(1)}$ as

$$A^{(2)} = -\ 2\zeta(\xi + \eta)^{-1}, \quad B^{(2)} = -\ 2\zeta^{-1}(\xi + \eta)^{-1}.$$

Then, by the procedure given in Proposition 4.1.4, we get

$$f^{(2)} = \gamma^{(2)}|(\zeta + 1)(\zeta - 1)^{-1}|^2, \quad \psi^{(2)} = 0,$$

where $\gamma^{(2)}$ is a positive constant. This is the result of the product transformation $\beta_- \circ \beta_+$.

Remark 4.2.1. Under the product transformation $\beta_+ \circ \beta_+$ or $\beta_- \circ \beta_-$, the resulting function f is the same as the original one within an additive constant. By using β_+ and β_- alternatively, a series of solutions $f^{(1)}$, $f^{(2)}$, ..., can be obtained.

We summarize the above discussions, taking Remark 4.2.1 into account to obtain

Proposition 4.2.2. *The succesive Bäcklund transformations* β_+ *and* β_- *yields* $f^{(N)}$ *as*

$$f^{(N)} = \gamma^{(N)}(\xi + \eta)|(\zeta - 1)(\zeta + 1)^{-1}|^{(N+1)/2} \quad (N\text{: odd}),$$

$$f^{(N)} = \gamma^{(N)}|(\zeta + 1)(\zeta - 1)^{-1}|^{(N+2)/2} \quad (N\text{: even}), \quad (4.2.3)$$

where $\gamma^{(N)}$ *in each cases are positive constants.*

It is not hard to express (4.2.3) in (ρ, z) coordinates. By the definition (3.2.1), we have

$$f^{(N)} = \gamma^{(N)}\rho|[\{(2\varepsilon + z)^2 + \rho^2\}^{1/2}$$
$$\mp (2\varepsilon + z)]\rho^{-1}\}^{(N+1)/2} \quad (N\text{: odd}),$$

$$f^{(N)} = \gamma^{(N)}|[\{(2\varepsilon + z)^2 + \rho^2\}^{1/2}$$
$$\pm (2\varepsilon + z)]\rho^{-1}|^{(N+2)/2} \quad (N\text{: even}), \quad (4.2.4)$$

where the double signs are due to ζ.

We here set

$$\gamma^{(N)} = (4|\varepsilon|)^{\pm(N+1)/2} \quad (N\text{: odd}),$$

$$\gamma^{(N)} = (4|\varepsilon|)^{\mp(N+2)/2} \quad (N\text{: even}),$$

and take the limit $\varepsilon \to \infty$. Then the solutions (4.2.4) tend to

$$f^{(N)} = \rho^{1\pm(N+1)/2} \quad (N\text{: odd}),$$

$$f^{(N)} = \rho^{\mp(N+2)/2} \qquad (N\text{: even}). \qquad (4.2.5)$$

Finally, we write out the metric for (4.2.5). Substituting (4.2.5) and $\omega^{(N)} = 0$ into (3.1.2), the remaining unknown function $\Gamma(\rho, z)$ can be determined. Then we have concrete metrices

$$ds^2 = -\rho^{1\pm(N+1)/2}\, dt^2 + \rho^{(N+3)(N-1)/8}(d\rho^2 + dz^2) + \rho^{1\pm(N+1)/2}\, d\phi^2 \qquad (N\text{: odd}),$$

$$ds^2 = -\rho^{\mp(N+2)/2}\, dt^2 + \rho^{(N+2)(N+2\pm 4)/8}(d\rho^2 + dz^2) + \rho^{2\pm(N+2)/2}\, d\phi^2 \qquad (N\text{: even}).$$

By suitable coordinate transformations, these are reduced, respectively, to

$$ds^2 = -\rho^{2\lambda}\, dt^2 + d\rho^2 + \rho^{2\mu}\, dz^2 + \rho^{2\nu}\, d\phi^2 ,$$

$$\lambda = 4(2 \pm N \pm 1)/C, \qquad \mu = (N^2 + 2N - 3)/C,$$

$$\nu = 4(2 \mp N \mp 1)/C, \qquad C = N^2 + 6N + 2 \qquad (N\text{: odd}),$$

$$\lambda = \mp 4(N + 2)/D, \qquad \mu = \{N^2 + (4 \pm 4)N + 4 \pm 8\}/D,$$

$$\nu = 4(4 \pm N \pm 2)/D, \qquad D = N^2 + (4 \pm 4)N + 4 \pm 24$$

$$(N\text{: even}). \qquad (4.2.6)$$

Since $\lambda + \mu + \nu = 1$, $\lambda\mu + \mu\nu + \nu\lambda = 0$ in both cases, the metrices (4.2.6) are the polynomial solutions discoverd by Kasner [4]. Thus the solutions (4.2.4) are new generalizations of the Kasner solutions.

4.3. Riemann-Hilbert transformation

A second transformation theory of the Einstein equations is an application of the Riemann-Hilbert problem. For the chiral field equations, such kind of transformation theory is set up by the method of solving an *algebraic* Riemann-Hilbert problem [9]. Following this method, Belinsky and Zakharov [1] have integrated the stationary axially symmetric Einstein equations to obtain a Kerr-NUT solution. However, their framework is limitted within the construction of spacial solutions.

It is important to develop a transformation theory concerning a *full* Riemann-Hilbert problem. In this direction, recently Hauser and Ernst [3] have studied a homogeneous Riemann-Hilbert problem associated with a set of linear differential equations. They have presented a transformation from a fundamental solution to another in a rather complicated manner. We call it *the Riemann-Hilbert transformation*. To avoid the difficulties due to complicated spectral parameter, we propose an alternative approach to the Riemann-Hilbert problem of the Einstein equations. This is the purpose of this section.

We discuss the Einstein equations (3.1.8) and (3.1.9) over a complexified space. Define the 2×2 matrix $P = P(\rho, z)$ by

$$P = \frac{1}{f}\begin{pmatrix} 1 & \psi \\ \psi & f^2 + \psi^2 \end{pmatrix} . \tag{4.3.1}$$

Note here $\det P = 1$. It can be proved that the field equations (3.1.8) and (3.1.9) are equivalent to

$$\partial_\rho(\rho\partial_\rho P\cdot P^{-1}) + \partial_z(\rho\partial_z P\cdot P^{-1}) = 0. \tag{4.3.2}$$

We call the factrization (4.3.1) of P "dual" to (3.1.4) of Q, in the sense that Equations (4.3.2) are the same as Equations (3.1.3).

Let us analytically continue $P(\rho, z)$ into the complex space $(y, \bar{y}, z)$, where $\bar{y}$ denotes a variable independent of the complex conjugate y^* of y and $\rho^2 = y\bar{y}$. Real Euclidean space is specified by $y = \bar{y}^*$, and consequently,

$$2y^{-1}\partial_{\bar{y}} = 2\bar{y}^{-1}\partial_y = \rho^{-1}\partial_\rho . \tag{4.3.3}$$

We note further that a solution of (4.3.2) is given by a solution $P(\rho, z)$ with $\rho^2 = y\bar{y}$ of the equations

$$\partial_{\bar{y}}(\partial_y P \cdot P^{-1}) + \partial_z(\partial_z P \cdot P^{-1}) = 0, \qquad (4.3.4)$$

where $P \in SL(2, \mathbb{C})$. In addition, Equations (4.3.4) can be obtained as Eular-Lagrange equations for the Lagrangian density

$$L = \mathrm{Tr}\{\partial_y P \partial_{\bar{y}}(P^{-1}) + \partial_z P \partial_z(P^{-1})\}. \qquad (4.3.5)$$

We consider variational equations for (4.3.4) by the infinitesimal transformation

$$P \to P(1 + \Lambda), \qquad (4.3.6)$$

where Λ is a 2×2 matrix function of y, $\bar{y}$ and z. We show

Lemma 4.3.1. *If the infinitesimal transformation satisfies*

$$(\partial_y \partial_{\bar{y}} + \partial_z^{\,2})\Lambda + [P^{-1}\partial_{\bar{y}} P, \partial_y \Lambda]$$
$$+ [P^{-1}\partial_z P, \partial_z \Lambda] = 0, \qquad (4.3.7)$$

then Λ *gives a symmetry of* (4.3.4).

Proof. We set $P' = P(1 + \Lambda)$. Substituting P' and $P'^{-1} = (1 - \Lambda)P^{-1}$ into (4.3.4) and neglecting higher order terms in Λ, we have

$$\partial_{\bar{y}}(\partial_y P' \cdot P'^{-1}) + \partial_z(\partial_z P' \cdot P'^{-1})$$

$$\cong \partial_{\bar{y}}(\partial_y P \cdot P^{-1}) + \partial_z(\partial_z P \cdot P^{-1}) + P\{(\partial_y \partial_{\bar{y}} + \partial_z{}^2)\Lambda$$
$$+ [P^{-1}\partial_{\bar{y}} P, \partial_y \Lambda] + [P^{-1}\partial_z P, \partial_z \Lambda]\}P^{-1}.$$

This proves the lemma, □

Using Lemma 4.3.1, we can introduce a sequence of nonlocal symmetries $\{S_0 = \Lambda_0, S_1, S_2, \ldots\}$, where Λ_0 is a constant matrix sufficiently close to 0, as we will prove. We show the following proposition.

Proposition 4.3.2. *A sequence* $\{S_n\}$, $n \geq 0$, *defined by* $S_0 = \Lambda_0$ *and*

$$\partial_y S_{n+1} + \partial_z S_n + [P^{-1}\partial_z P, S_n] = 0,$$
$$\partial_z S_{n+1} - \partial_{\bar{y}} S_n - [P^{-1}\partial_{\bar{y}} P, S_n] = 0 \tag{4.3.8}$$

is an infinite number of nonlocal symmetries of (4.3.4).

Proof. We take cross derivatives of S_{n+1}. The integrability condition gives

$$(\partial_y \partial_{\bar{y}} + \partial_z{}^2)S_n + [P^{-1}\partial_{\bar{y}} P, \partial_y S_n] + [P^{-1}\partial_z P, \partial_z S_n]$$
$$+ [\partial_y(P^{-1}\partial_{\bar{y}} P) + \partial_z(P^{-1}\partial_z P), S_n] = 0.$$

The last term varishes if P is a solution of (4.3.4). Then each S_n, $n \geq 0$, satisfies (4.3.7) and is nonlocal symmetry of (4.3.4) (see Section 2.4). This completes the proof. □

We set a generating function $S(\zeta)$ as

$$S(\zeta) = \sum_{n=0}^{\infty} S_n \zeta^n, \tag{4.3.9}$$

where ζ is a complex constant. From (4.3.8), we have

$$D_1 S(\zeta) + \zeta[P^{-1}\partial_z P, S(\zeta)] = 0,$$

$$D_2 S(\zeta) - \zeta[P^{-1}\partial_{\bar{y}} P, S(\zeta)] = 0, \tag{4.3.10}$$

where D_1 and D_2 are differential operators

$$D_1 = \partial_y + \zeta\partial_z, \quad D_2 = \partial_z - \zeta\partial_{\bar{y}}. \tag{4.3.11}$$

Let $V(\zeta) = V(y, \bar{y}, z; \zeta)$ be a 2×2 matrix function satisfying

$$D_k V(\zeta) = 0, \quad V(0) = \Lambda_0, \quad k = 1, 2. \tag{4.3.12}$$

We define the following 2×2 matrix function $Y(\zeta) = Y(y, \bar{y}, z; \zeta)$ under the condition

$$S(\zeta) = P^{-1}Y(\zeta)V(\zeta)Y(\zeta)^{-1}P. \tag{4.3.13}$$

Providing that $Y(\zeta)$ be analytic near $\zeta = 0$, we may take

$$Y(0) = P, \tag{4.3.14}$$

because of $S(0) = V(0) = \Lambda_0$. Then we have

Proposition 4.3.3. *Equations* (4.3.4) *are equivalent to the compatibility condition of the linear differential equations*

$$D_1 Y(\zeta) = \partial_y P \cdot P^{-1} Y(\zeta),$$

$$D_2 Y(\zeta) = \partial_z P \cdot P^{-1} Y(\zeta). \tag{4.3.15}$$

Proof. Substituting (4.3.13) with(4.3.12) into (4.3.10), we obtain

$$[Y(\zeta)^{-1}(D_1 Y(\zeta) - \partial_y P \cdot P^{-1} Y(\zeta)),\ V(\zeta)] = 0,$$

$$[Y(\zeta)^{-1}(D_2 Y(\zeta) - \partial_z P \cdot P^{-1} Y(\zeta)),\ V(\zeta)] = 0,$$

for arbitrary ζ and $V(\zeta)$. These equations are consistent (4.3.15). The compatibility condition of (4.3.15) reads

$$[D_1 - \partial_y P \cdot P^{-1},\ D_2 - \partial_z P \cdot P^{-1}]$$

$$= -\zeta\{\partial_{\bar{y}}(\partial_y P \cdot P^{-1}) + \partial_z(\partial_z P \cdot P^{-1})\}.$$

This proves the proposition. $\square$

The system (4.3.15) is an inverse scattering formula for (4.3.4). As an application of Proposition 4.3.3, we have a series of nonlocal conservation laws. Expand $Y(\zeta)$ in a power series of ζ,

$$Y(\zeta) = \sum_{n=0}^{\infty} Y_n \zeta^n. \tag{4.3.16}$$

Then from (4.3.15) the sequence $\{Y_n\}$, $n \geq 0$, satisfies the equations $Y_0 = P$ and

$$\partial_y Y_{n+1} + \partial_z Y_n - \partial_y P \cdot P^{-1} Y_n = 0,$$
$$\partial_z Y_{n+1} - \partial_{\bar{y}} Y_n - \partial_z P \cdot P^{-1} Y_n = 0, \tag{4.3.17}$$

which derives an infinite number of nonlocal conservation laws

$$\partial_z(\partial_z Y_n - \partial_y P \cdot P^{-1} Y_n)$$
$$+ \partial_y(\partial_{\bar{y}} Y_n + \partial_z P \cdot P^{-1} Y_n) = 0. \tag{4.3.18}$$

We proceed to discuss the Riemann-Hilbert problem for (4.3.15). Let C be a closed analytic curve in the ζ-plane encircling the origin, and C_+ and C_- be the inside and outside of C, respectively. Supposing the curve C be so small that $Y(\zeta)$ is analytic in $C \cup C_+$. we consider the following Riemann-Hilbert problem of finding matrices $X_+(\zeta)$ and $X_-(\zeta)$ such that

$$X_-(\zeta') = X_+(\zeta')H(\zeta'), \qquad \zeta' \in C,$$
$$H(\zeta) = Y(\zeta)u(\zeta)Y(\zeta)^{-1}, \tag{4.3.19}$$

where $X_\pm(\zeta)$ are invertible and analytic in $C \cup C_\pm$, respectively, and $u(\zeta)$ is an $SL(2, \mathbb{C})$ matrix annihilated by

D_1 and D_2, that is,

$$D_k u(\zeta) = 0, \quad k = 1, 2. \tag{4.3.20}$$

We assume that this problem has solutions with the normalization condition

$$X_-(\infty) = 1. \tag{4.3.21}$$

Following Hauser and Ernst [3], we set

$$Y'(\zeta) = \begin{cases} X_+(\zeta)Y(\zeta) & \text{in } C_+, \\ X_-(\zeta)Y(\zeta)u(\zeta)^{-1} & \text{in } C_-. \end{cases} \tag{4.3.22}$$

This is the Riemann-Hilbert transformation induced from $u(\zeta)$. Then we obtain a transformation theorem as follows.

Theorem 4.3.4. *Suppose* $P = Y(0)$ *be a solution of* (4.3.4). *The matrix* P' *defined by* $P' = Y'(0)$ *is another solution.*

Proof. We operate (4.3.19) with D_k, and use (4.3.15) and (4.3.20) to get

$$D_1 X_+ \cdot X_+^{-1} + X_+ \partial_y P \cdot P^{-1} X_+^{-1}$$

$$= D_1 X_- \cdot X_-^{-1} + X_- \partial_y P \cdot P^{-1} X_-^{-1},$$

$$D_2 X_+ \cdot X_+^{-1} + X_+ \partial_z P \cdot P^{-1} X_+^{-1}$$

$$= D_2 X_- \cdot X_-^{-1} + X_- \partial_z P \cdot P^{-1} X_-^{-1},$$

where we write $X_\pm = X_\pm(\zeta')$, $\zeta' \in C$, for simplicity. Let us set $X(\zeta) = X_\pm(\zeta)$ in $C_\pm$. Therefore, B_k, $k = 1, 2$, defined by

$$B_1 = D_1 X(\zeta) \cdot X(\zeta)^{-1} + X(\zeta) \partial_y P \cdot P^{-1} X(\zeta)^{-1},$$

$$B_2 = D_2 X(\zeta) \cdot X(\zeta)^{-1} + X(\zeta) \partial_z P \cdot P^{-1} X(\zeta)^{-1}$$

are constant on the ζ-plane (Liouville's theorem). Using Equations (4.3.15) and (4.3.22), we have

$$D_k Y'(\zeta) = B_k Y'(\zeta), \quad k = 1, 2. \tag{4.3.23}$$

The compatibility condition of (4.3.23) gives

$$\partial_z B_1 - \partial_y B_2 + [B_1, B_2] = 0,$$

$$\partial_{\bar{y}} B_1 + \partial_z B_2 = 0. \tag{4.3.24}$$

On the other hand, we set $\zeta = 0$ in (4.3.23) to get

$$B_1 = \partial_y P' \cdot P'^{-1}, \quad B_2 = \partial_z P' \cdot P'^{-1}.$$

Then the first equations of (4.3.24) are satisfied automatically. The second equations are equivalent to (4.3.4).

We next verify that $\det P' = 1$. Using $\det u(\zeta) = 1$ and (4.3.19), we obtain

$$\det X_+(\zeta') = \det X_-(\zeta'), \quad \zeta' \in C.$$

Therefore, det $X(\zeta)$ is an entire function of ζ. Since $X(\zeta)$ is regular at infinity, det $X(\zeta)$ is constant. From the condition (4.3.21),

$$\det X(\zeta) = 1.$$

It follows that if $\det P = 1$, then $\det P' = \det Y'(0) = 1$. This completes the proof. □

The totality of the Riemann-Hilbert transformations (4.3.22) induced from $u_1(\zeta)$, $u_2(\zeta)$, ..., is closed under the composition, which is guaranteed by the following proposition.

Proposition 4.3.5. *Suppose that* $D_k u_1(\zeta) = D_k u_2(\zeta) = 0$, $k = 1, 2$, $\det u_1(\zeta) = \det u_2(\zeta) = 1$, *and* $u_1(\zeta)$ *and* $u_2(\zeta)$ *designate the transformation of* $Y(\zeta)$ *into* $Y_1(\zeta)$ *and of* $Y_1(\zeta)$ *into* $Y_2(\zeta)$, *respectively. Then* $u_2(\zeta)u_1(\zeta)$ *designates that of* $Y(\zeta)$ *into* $Y_2(\zeta)$.

Proof. From the definition (4.3.22), we have

$$Y_1(\zeta) = X_{1-}(\zeta)Y(\zeta)u_1(\zeta)^{-1},$$

$$Y_2(\zeta) = X_{2-}(\zeta)Y_1(\zeta)u_2(\zeta)^{-1}.$$

These equations give a composition law of the Riemann-Hilbert transformations

$$Y_2(\zeta) = X_{2-}(\zeta) X_{1-}(\zeta)\, Y(\zeta)\, (u_2(\zeta) u_1(\zeta))^{-1}. \qquad (4.3.25)$$

This proves the proposition. □

From Proposition 4.3.5, it is shown that each Riemann-Hilbert transformation gives a germ of a transformation group. Obviously the group germ depend on the choice of the circle C.

We direct attention to the condition $D_k u(\zeta) = 0$. This implies that $u(\zeta)$ is an arbitrary function of ζ and υ, the latter being defined by

$$\upsilon = \zeta y - z - \zeta^{-1}\bar{y}. \qquad (4.3.26)$$

The quantity υ also emerges in the theory of Yang-Mills-Higgs monopole solutions (see [8]).

CHAPTER V

SYMMETRIES OF THE EINSTEIN EQUATIONS

The *static* axially symmetric vacuum gravitational fields were completely investigated by Weyl in 1917. Afterward, the *stationary* field equations have been studied from a mathematical and physical points of view. As was mentioned in Chapter I, two ways have been found to exact solutions of the field equations. One is based on the soliton theory (see Chapter III and IV). The other is the group theoretical approach based on the symmetry of the Einstein equations.

Geroch group gives the most fundamental idea to develop the latter solution-generating method. Combining an $SL(2, \mathbb{R})$ coordinate transformation group with a hidden $SL(2, \mathbb{R})$ symmetry group, Geroch [6] presented an infinite-dimensional transformation group. The idea was pushed forward by Kinnersley and Chitre [11, 12], who constructed a useful representation of the infinitesimal generators by introducing an infinite hierarchy of potentials. In Section 5.1, after reviewing the Kinnersley-Chitre theory of the Geroch group, it is shown that the infinitesimal transformations of the Geroch group form an infinite-dimensional Lie algebra, which is isomorphic the the graded Lie algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$.

Kinnersley-Chitre [13, 14] and Hoenselaers-Kinnersley-Xanthopoulos [8] set up methods for exponentiating the infinitesimal transformations which are corresponding to *abelian* subgroups of the Geroch group. Thus the commutative transformations they found can be related to the Bäcklund transformation (cf. Theorem of permutability (2.2.2)). In Section 5.2, we give a Bäcklund transformation without exponentiating the infinitesimal transformations. Two types of internal symmetries of the field equations are combined to construct the Bäcklund transformation which indispensable for our theory. Having defined a special function which relates the field equations to a linear differential equation, we derive a family of exact solutions in Section 5.3. The resulting solutions take the form of the ratio of determinants. Finally, in Section 5.4, as an application of the Bäcklund transformation, we give by solving linear differential equations concrete solutions which correspond to the ansatze.

5.1. Geroch's transformation group

The Einstein gravitational field equations are manifestly covariant under the coordinate transformations. There is an $SL(2, \mathbb{R})$ invariance group which arises from linear transformations of the two ignorable coordinates

t and ϕ. On the other hand, the Ernst form of the Einstein equations also admits another $SL(2, \mathbb{R})$ symmetry group called NUT group. Unlike the coordinate transformation group, this symmetry group is *hidden* and makes a physical change in the solutions. One of the generators is called the *Ehlers transformation* [5]. It should be noted that the actions of these two *finite*-dimensional symmetry groups do not commute. Geroch payed attention to this fact and produced an *infinite*-dimensional transformation group as a product group.

In this section, we first formulate the Geroch group without using tensors. Next, we give a precise description of Kinnersley-Chitre's infinitesimal transformations and discuss the structure of a graded Lie algebra.

The stationary axially symmetric vacuum gravitational field equations (3.1.3) are invariant under the coordinate transformations

$$g: \begin{pmatrix} t \\ \phi \end{pmatrix} \to \begin{pmatrix} t' \\ \phi' \end{pmatrix} = g \begin{pmatrix} t \\ \phi \end{pmatrix}, \tag{5.1.1}$$

where g is an $SL(2, \mathbb{R})$ constant matrix. The three independent generators can be expressed as

$$g_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad g_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad g_c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix},$$

where a, b and c are nonzero real parameters. The metric coefficients $Q = (q_{\mu\nu})$ are transformed into $Q' = (q'_{\mu\nu})$ as follows:

$$g:\ Q \to Q' = {}^t g^{-1} Q g^{-1}, \tag{5.1.2}$$

where the superscript t stands for the transposition. We refer to this $SL(2, \mathbb{R})$ symmetry group as G. For example, the infinitesimal transformation of g_b is expressed in the factrization of Q as

$$\dot{g}_b:\ \begin{pmatrix} f \\ \omega \end{pmatrix} \to \begin{pmatrix} f' \\ \omega' \end{pmatrix} \cong \begin{pmatrix} f - 2bf\omega \\ \omega + b(\omega^2 + \omega^2 f^{-2}) \end{pmatrix}.$$

We write by $g^{(0)}$ the Lie algebra of G.

Next, we consider the dual factrization of P, (see (4.3.1.)). The field equations (4.3.2) are invariant under the transformations

$$h:\ P \to P' = hP\,{}^t h, \tag{5.1.3}$$

where h is an $SL(2, \mathbb{R})$ constant matrix. This hidden symmetry group of the Einstein equations is called the NUT group and will be referred to H. Let us set three generator of H as

$$h_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad h_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad h_c = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix},$$

where a, b and c are real parameters. Then the actions of H in the factrization (4.3.1) take the form

$$h_a: \begin{pmatrix} f \\ \psi \end{pmatrix} \to \frac{1}{1 + 2a\psi + a^2(f^2 + \psi^2)} \begin{pmatrix} f \\ \psi + a(f^2 + \psi^2) \end{pmatrix} ,$$

$$h_b: \begin{pmatrix} f \\ \psi \end{pmatrix} \to \begin{pmatrix} f \\ \psi + b \end{pmatrix} ,$$

$$h_c: \begin{pmatrix} f \\ \psi \end{pmatrix} \to \begin{pmatrix} c^2 f \\ c^2 \psi \end{pmatrix} . \tag{5.1.4}$$

The first of these is the Ehlers transformation [5], which exchanges the Schwarzchild solution depending on one parameter into the Taub-NUT solutions with two parameters, for example. The second and third are a gauge and a scale transformation, respectively. The infinitesimal Ehlers transformation

$$h_a: \begin{pmatrix} f \\ \psi \end{pmatrix} \to \begin{pmatrix} f + 2af\psi \\ \psi - a(f^2 - \psi^2) \end{pmatrix} \tag{5.1.5}$$

is important in the subsequent discussions.

Kinnersley [11] introduced a generalization of the Ernst potential by

$$E = Q + i\Psi, \tag{5.1.6}$$

where the 2×2 matrix $\Psi = (\psi_{\mu\nu})$ is given by

$$\nabla\Psi = \rho^{-1}Q\sigma\tilde{\nabla}Q, \quad \text{with}$$

$$\nabla = \begin{pmatrix} \partial_\rho \\ \partial_z \end{pmatrix}, \quad \tilde{\nabla} = \begin{pmatrix} \partial_z \\ -\partial_\rho \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The (1, 1) component of E, $q_{11} + i\psi_{11}$, corresponds to the useal Ernst potential (3.1.10). Let $v^{(0)}$ be an element of $sym(2, \mathbb{R})$, the algebra of 2×2 real matrices whose bracket operation is defined by $[v, v'] = v\sigma v' - v'\sigma v$. By the isomorphism $sl(2, \mathbb{R}) \cong sym(2, \mathbb{R})$ established by σ, we identify $v^{(0)}$ with an element of $sl(2, \mathbb{R})$. Then the group G acting on the space of E's by ${}^tg^{-1}Eg^{-1}$ has an infinitesimal generator $v^{(0)}$,

$$v^{(0)}: E \to E + v^{(0)}\sigma E - E\sigma v^{(0)}. \tag{5.1.8}$$

The action of the infinitesimal Ehlers transformation induced by h_a on the potential E is shown to take the form

$$\dot{h}_a: E \to E + iv_a\sigma N - iN\sigma v_a + iE\sigma(v_a\sigma E - E\sigma v_a), \tag{5.1.9}$$

$$\text{with} \quad \nabla N = E^\dagger\sigma\nabla E, \quad v_a = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix},$$

where $\dagger$ denotes the hermitian conjugate. The proof of (5.1.9) is accomplished by tedious but straightforward calculation [11, 12]. The infinitesimal transformation

(5.1.9) has only one parameter a. We denote (5.1.9) by $v_a\zeta^{-1}$, where ζ is a real parameter.

Combining the action of G with (5.1.9), we define an algebra $g^{(1)}$ by

$$g^{(1)} = \{[\sigma v_x\zeta^{-1}, \dot{g}_x]\} \oplus \{\sigma v_y\zeta^{-1}\}, \tag{5.1.10}$$

where $x, y = a, b, c$. We note that $g^{(1)}$ is isomorphic to $sym(2, \mathbb{R}) \otimes \zeta^{-1}$. We can identify $v^{(1)}\zeta^{-1}$, $v^{(1)} \in sym(2, \mathbb{R})$, with an element of $g^{(1)}$. The infinitesimal transformations $v^{(1)}\zeta^{-1}$ with three parameters is proved to take the form [12]

$$\begin{aligned} v^{(1)}\zeta^{-1}\colon E \to E + i\dot{v}^{(1)}\sigma N - iN\sigma v^{(1)} \\ + iE\sigma(v^{(1)}\sigma E + E\sigma v^{(1)}). \end{aligned} \tag{5.1.11}$$

Furthermore, we set

$$g^{(k+1)} = [g^{(k)}, g^{(1)}], \quad k \geq 1. \tag{5.1.12}$$

Then the infinitesimal transformations of the Geroch group [6] can be given by the action of $g^{(k)}$'s on the E-space. To represent $g^{(k)}$, Kinnersley and Chitre [12] introduced an infinite number of potentials $N^{(m,n)}$, $m \geq 0$, $n \geq 1$, by

$$E^{(0)} = i\sigma, \quad E^{(1)} = E, \quad N^{(1,1)} = N,$$

$$E^{(n+1)} = iN^{(1,n)} + iE\sigma E^{(n)},$$

$$\nabla N^{(m,n)} = E^{(m)\dagger}\sigma\nabla E^{(n)}. \tag{5.1.13}$$

We observe from the definition that $g^{(k)}$ is isomorphic to $sym(2, \mathbb{R}) \otimes \zeta^{-k}$ as a vector space. Under this isomorphism, $v^{(k)}\zeta^{-k}$ can be thought of as an infinitesimal transformation which acts on $N^{(m,n)}$ as follows:

$$\begin{aligned} v^{(k)}\zeta^{-k}\colon N^{(m,n)} \to N^{(m,n)} &+ v^{(k)}\sigma\, N^{(m+k,n)} \\ &- N^{(m,n+k)}\sigma v^{(k)} \\ &- \sum_{j=1}^{k} N^{(m,j)}\sigma v^{(k)}\sigma N^{(k-j,n)}. \end{aligned} \tag{5.1.14}$$

The proof of (5.1.14) [12] is given by induction together with the recursion relations

$$N^{(m,n+1)} - N^{(m+1,n)} = iN^{(m,1)}\sigma E^{(n)}.$$

From (5.1.14), one can prove

$$\begin{aligned} &[\![v^{(k)}\zeta^{-k}, v^{(l)}\zeta^{-l}]\!] \\ &= [v^{(k)}, v^{(l)}]\zeta^{-(k+l)}, \quad k,l \geq 0, \end{aligned} \tag{5.1.15}$$

where the bracket $[\![\ ,\ ]\!]$ indicates the commutator of the infinitesimal operators acting on the space of $N^{(m,n)}$. Equation (5.1.15) means that the left-hand side is equal to the infinitesimal transformation induced by the right-hand side.

Let us define a class of infinitesimal gauge transformations $g^{(-1)}$ being isomorphic to $sym(2,\mathbb{R})\otimes\zeta$ by

$$v^{(-1)}\zeta\colon\ E \to E + iv^{(-1)},\qquad v^{(-1)} \in sym(2,\mathbb{R}).$$

In general, we set

$$g^{(-k)} = [g^{(-k-1)},\ g^{(-1)}],\qquad k \geq 1. \tag{5.1.16}$$

Then we have

$$\begin{aligned} v^{(-k)}\zeta^k\colon\ N^{(m,n)} \to N^{(m,n)} &- v^{(-k)}\sigma N^{(m-k,n)} \\ &- N^{(m,n-k)}\sigma v^{(k)} \\ &- \sum_{j=0}^{k-1} \delta_{m,j} v^{(-k)} \delta_{k-j,n}, \end{aligned} \tag{5.1.17}$$

where δ_{ij} are Kronecker's delta. Here the potentials with negative indices are defined by

$$N^{(m,-m)} = -N^{(-m,m)} = \sigma,$$

and the others are all equal to 0. Then the commutation relations

$$\begin{aligned} &[\![v^{(-k)}\zeta^k,\ v^{(-l)}\zeta^l]\!] \\ &= [v^{(-k)},\ v^{(-l)}]\zeta^{k+l},\qquad k,\ l \geq 0, \end{aligned} \tag{5.1.18}$$

are proved to be true (see the notice which follows (5.1.15)). We now obtain the following main theorem.

Theorem 5.1.1. *Let us define the algebra by*

$$g = \bigoplus_{k=-\infty}^{\infty} g^{(k)},$$

where $g^{(k)}$ *are defined by* (5.1.10, 12, 16). *Then* g *is isomorphic to the graded Lie algebra*

$$sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}], \tag{5.1.19}$$

the graded algebra of $sl(2, \mathbb{R})$*-valued Laurent series.*

Proof. From (5.1.14) and (5.1.17), we have, after calculation, the commutation relations of infinitesimal operators,

$$[\![v^{(k)}\zeta^{-k}, v^{(l)}\zeta^{-l}]\!]$$

$$= [v^{(k)}, v^{(l)}]\zeta^{-(k+l)} \tag{5.1.20}$$

for any integers k, l. This proves the theorem. □

What this theorem means is that the infinitesimal transformations of the Geroch group form an infinite-dimensional Lie algebra (5.1.19). This graded algebra of $sl(2, \mathbb{R})$-valued series in ζ is one of the simplest affine Lie algebras which have been studied recently by many mathematicians [9, 15].

5.2. Symmetries and Bäcklund transformation

Pushing the approach discussed in Section 5.1 forward, Kinnersley and Chitre [13, 14] and Hoenselaers, Kinnersley and Xanthopoulos [8] have exponentiated several special classed of the infinitesimal transformations of the Geroch group. These transformations have the important property of preserving asymptotic flatness of solutions. However, in general no finite transformation of the Geroch group which generates new stationary axially symmetric vacuum solutions has been known until now. In this section, we gain a deeper insight into the internal symmetries of the field equations discussed in the previous section in the name of hidden symmetry, and present a *finite* transformation of the Geroch group as a Bäcklund transformation.

The field equations (4.3.2) are invariant under the $sl(2, \mathbb{R})$ rotation (5.1.3). Particularly, we take the element h of $sl(2, \mathbb{R})$:

$$h = \begin{pmatrix} 0 & \gamma^{-1} \\ -\gamma & 0 \end{pmatrix}, \quad \gamma \neq 0. \tag{5.2.1}$$

Using the h_γ, we obtain the following lemma.

Lemma 5.2.1. *If* (f, ψ) *with* $f^2 + \psi^2 \neq 0$ *satisfies the field equations* (3.1.8) *and* (3.1.9), *then* (f', ψ') *defined by*

$$f' = \gamma^2 f(f^2 + \psi^2)^{-1},$$

$$\psi' = -\gamma^2 \psi(f^2 + \psi^2)^{-1} \qquad (5.2.2)$$

satisfies the same field equations.

Proof. Applying the $SL(2, \mathbb{R})$ rotation (5.1.3), we have

$$P' = \frac{1}{f}\begin{pmatrix} \gamma^{-2}(f^2 + \psi^2) & -\psi \\ -\psi & \gamma^2 \end{pmatrix}.$$

Since P' can be factorized as

$$P' = \frac{1}{f'}\begin{pmatrix} 1 & \psi' \\ \psi' & f'^2 + \psi'^2 \end{pmatrix},$$

new solution (f', ψ') of (3.1.8) and (3.1.9) is given by (5.2.2). This proves the lemma. □

We call (5.2.2) the transformation γ after the work of Corrigan, Fairlie, Yates and Goddard [3]. They have found a Bäcklund transformation of the $SU(2)$ self-dual

Yang-Mills equations in gauge theory. It is worth mentioning that the transformation γ is an involution: $\gamma^2 = \text{identity}$. This transformation is an internal symmetry.

Next, we consider another internal symmetry of (3.1.8) and (3.1.9). Let us recall the factrizations (3.1.4) and (4.3.1). Then we have

Lemma 5.2.2. *There is a mapping under which Equations* (3.1.5) *and* (3.1.6) *transform to Equations* (3.1.8) *and* (3.1.9), *respectively, which we denote by* I:

$$I: \begin{pmatrix} f \\ \omega \end{pmatrix} \to \begin{pmatrix} \rho f^{-1} \\ i\psi \end{pmatrix} . \tag{5.2.3}$$

The proof is immediate by inspection. This operation was introduced originally by Neugebauer and Kramer [18].

The metric coefficients f and ω and the twist potential ψ are real functions of ρ and z, while the mapping I defined by (5.2.3) transforms real potentials to complex ones. Hence, it seems natural to treat ψ as analytically continued function into complex space. We can show

Lemma 5.2.3. *Let* (f, ψ) *be a solution of* (3.1.8) *and* (3.1.9). *Then* (f', ψ') *determined by*

$$f' = \rho f^{-1},$$

$$\partial_\rho \psi' = -i\rho f^{-2} \partial_z \psi, \quad \partial_z \psi' = i\rho f^{-2} \partial_\rho \psi \tag{5.2.4}$$

is also a solution.

Proof. From the definition of the twist potential (3.1.7), we have

$$\partial_\rho \omega = -\rho f^{-2} \partial_z \psi, \quad \partial_z \omega = \rho f^{-2} \partial_\rho \psi. \tag{5.2.5}$$

Suppose that (f, ω) transforms to $(\rho f'^{-1}, i\psi')$ under I. Then Equation (5.2.5) yields (5.2.4). This proves the lemma. □

We refer to the transformation (5.2.4) as β. It is obvious from (5.2.4) that β is an involution: $\beta^2 =$ identity, within an integration constant.

We now define a product transformation α:

$$\alpha = \beta \circ \gamma \tag{5.2.6}$$

with the parameter γ in (5.2.1) equal to 1. Since $\beta \circ \gamma \neq \gamma \circ \beta$, α is not an involution: $\alpha^2 \neq$ identity, so that the α will generate a series of non-trivial solutions (f, ψ). The operation of α is interpreted as a Bäcklund transformation and is given by the theorem which we now state.

Theorem 5.2.4. *The transformation* α *acts on an initial solution* (f, ψ) *of* (3.1.8) *and* (3.1.9) *through*

$$f' = \rho f^{-1}(f^2 + \psi^2),$$

$$\partial_\rho \psi' = i\rho f^{-2}(f^2 + \psi^2)^2 \cdot \partial_z \{\psi(f^2 + \psi^2)^{-1}\},$$

$$\partial_z \psi' = -i\rho f^{-2}(f^2 + \psi^2)^2 \cdot \partial_\rho \{\psi(f^2 + \psi^2)^{-1}\}, \qquad (5.2.7)$$

where (f', ψ') *is another solution of* (3.1.8) *and* (3.1.9).

Proof. By substituting (5.2.2) with $\gamma = 1$ into (5.2.4), we derive (5.2.7) immediately. □

It should be noted that by applying α even times we can obtain real potentials from real ones. The Bäcklund transformation α is thought of as a finite transformation of the Geroch group discussed in the previous section. This is because the transformation α of (f, ψ) can be interpreted as a transformation of (f, ω) through the mapping I, and because, briefly speaking, the Geroch group is a group acting on the matrix Q expressed in terms of (f, ω).

Soliton theoretic treatment has presented several types of Bäcklund transformations for the Einstein equations. Recently, the relationship between these Bäcklund transformations and the Geroch group is investigated by Cosgrove [4].

He points out that the Bäcklund transformations proposed in Harriscn [7], Belinsky-Zakharov [2] and Section 4.1 are menbers of the Geroch group, while the transformation given in Neugebauer [16] includes the coordinate transformations and lies outside the Geroch group.

5.3. New family of exact solutions

As for the systematic generation of exact solutions, we call especial attention to the works of Kinnersley and Chitre [14] and Neugebauer [17]. The former has derived the Kerr-Tomimatsu-Sato family of asymptotic flat solutions. The latter has developed the method for finding a multi-soliton solution which describes a superposition of Kerr-NUT solutions.

On the other hand, in gauge theory, Corrigan, Fairlie, Yates and Goddard [3] have succeeded in integrating the *ansatze* of Atiyah and Ward [1], which yield the multi-instanton solutions of $SU(2)$ self-dual Yang-Mills equations. As a consequence it is shown that the gauge potentials are described by solutions of the four-dimensional Laplace equation.

In this section, first we define a special function which relates the field equations to a single linear differential equation. Then we show that there is a new family of exact solutions being outside the works of Kinnersley-Chitre [14] and Neugebauer [17]. The resulting solutions are described in terms of determinants.

We begin by introducing a function Δ by which one can reduce the field equations (3.1.8) and (3.1.9) to a linear equation. We have the following.

Theorem 5.3.1. *A solution of* (3.1.8) *and* (3.1.9) *is given by*

$$f = \partial_\rho \Delta, \quad \psi = \partial_z \Delta, \tag{5.3.1}$$

if $\Delta = \Delta(\rho, z)$ *is a solution of the linear equation*

$$(\partial_\rho^2 - \rho^{-1}\partial_\rho + \partial_z^2)\Delta = 0. \tag{5.3.2}$$

Proof. Substituting (5.3.1) into (3.1.8) and (3.1.9), we have

$$(\partial_\rho \Delta \partial_\rho - \partial_\rho^2 \Delta + \partial_z^2 \Delta + \rho^{-1}\partial_\rho \Delta)$$

$$\cdot (\partial_\rho^2 - \rho^{-1}\partial_\rho + \partial_z^2)\Delta = 0,$$

$$(\partial_\rho \Delta \partial_z - 2\partial_\rho \partial_z \Delta)(\partial_\rho^2 - \rho^{-1}\partial_\rho + \partial_z^2)\Delta = 0,$$

respectively. This proves the theorem. □

As a consequence of Theorem 5.3.1, the stationary axially symmetric Einstein equations are linearized via the function Δ. We call a pair (f, ψ) given by $(\partial_\rho \Delta, \partial_z \Delta)$ the ansatz P_2.

A similar situation is known for the case of Weyl's family of static axially symmetric vacuum solutions (see [10]). However, Weyl solutions take the form:

$$f = \exp(2\theta), \qquad \psi = 0,$$

where $\theta = \theta(\rho, z)$ satisfies the cylindrical Laplace equation

$$(\partial_\rho{}^2 + \rho^{-1}\partial_\rho + \partial_z{}^2)\theta = 0.$$

These heuristic considerations suggest that we should seek for a hierarchy of ansatze like P_2 by using the Bäcklund transformation α and the internal symmetries β and γ. We operate (f, ψ) of P_2 with the transformation β. We mean by (f', ψ') the transform of (f, ψ) under β. Substituting $f = \rho f'^{-1}$ and $\partial_z f = \partial_\rho \psi$ into the last equation of (5.2.4), we have $\partial_z f' = i\partial_z \psi'$, which is consistent with the second equation of (5.2.4). Therefore, neglecting the integration constant, we obtain

$$f' = i\psi'. \tag{5.3.3}$$

From this, we are allowed to write

$$(f', \psi') = (\Delta_0^{-1}, -i\Delta_0^{-1}). \tag{5.3.4}$$

We call the pair (f', ψ') of solutions expressible by (5.3.4) the ansatz P_1'. We then have $\beta: P_2 \to P_1'$.

The variable $\Delta_0 = \Delta_0(\rho, z)$ also satisfies a linear equation as Δ does.

Lemma 5.3.2. *If* Δ_0 *satisfies*

$$(\partial_\rho^{\ 2} + \rho^{-1}\partial_\rho + \partial_z^{\ 2})\Delta_0 = 0, \tag{5.3.5}$$

then $(f, \psi) = (\Delta_0^{-1}, -i\Delta_0^{-1})$ *is a solution of* (3.1.8) *and* (3.1.9).

Proof. Let us substitute $(f, \psi) = (\Delta_0^{-1}, -i\Delta_0^{-1})$ into the left-hand sides of (3.1.8) and (3.1.9). The results are

$$-\Delta_0^{-3}(\partial_\rho^{\ 2} + \rho^{-1}\partial_\rho + \partial_z^{\ 2})\Delta_0 = 0,$$

$$-\rho(\partial_\rho^{\ 2} + \rho^{-1}\partial_\rho + \partial_z^{\ 2})\Delta_0 = 0,$$

respectively. This proves the lemma. □

Next, we define a new ansatz P_2' such that α: $P_2' \to P_1'$. The ansatze P_2' and P_1' correspond to (f, ψ) and (f', ψ') in (5.2.7), respectively. We then prove the following proposition.

Proposition 5.3.3. *The ansatz* P_2' *is written as*

$$P_2' : (f, \psi) = \left(\frac{\rho\Delta_0}{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}} , \frac{\Delta_1}{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}} \right), \tag{5.3.6}$$

where $\Delta_1 = \Delta_1(\rho, z)$ *is introduced by*

$$\partial_\rho \Delta_1 = -\rho \partial_z \Delta_0, \qquad \partial_z \Delta_1 = \rho \partial_\rho \Delta_0. \tag{5.3.7}$$

Proof. Substituting (5.3.4) into (5.2.7), we have

$$f^2 (f^2 + \psi^2)^{-1} = \rho\Delta_0, \tag{5.3.8}$$

$$\partial_\rho \{\psi(f^2 + \psi^2)^{-1}\} = \rho \partial_z \Delta_0,$$

$$\partial_z \{\psi(f^2 + \psi^2)^{-1}\} = -\rho \partial_\rho \Delta_0. \tag{5.3.9}$$

From the definition (5.3.7), Equations (5.3.9) read

$$\partial_\rho \{\psi(f^2 + \psi^2)^{-1}\} = -\partial_\rho \Delta_1,$$

$$\partial_z \{\psi(f^2 + \psi^2)^{-1}\} = -\partial_z \Delta_1.$$

Then we can integrate these equations to obtain

$$\psi(f^2 + \psi^2)^{-1} = -\Delta_1. \tag{5.3.10}$$

By solving the algebraic equations (5.3.8) and (5.3.10), we express the ansatz P_2' explicitely by (5.3.6). This completes the proof. □

We remark that the integrability condition for Δ_0 in (5.3.7) implies

$$(\partial_\rho{}^2 - \rho^{-1}\partial_\rho + \partial_z{}^2)\Delta_1 = 0. \tag{5.3.11}$$

We have the following corollary of Proposition 5.3.3.

Corollary 5.3.4. *The ansatz* P_2 *defined by takes the form*

$$P_2;\ (f,\ \psi) = (\rho\Delta_0,\ \Delta_1), \tag{5.3.12}$$

where Δ_0 *and* Δ_1 *are given by* (5.3.5) *and* (5.3.7), *respectively.*

Proof. From the transformations $\alpha\colon P_2' \to P_1'$ and $\beta\colon P_2 \to P_1'$ together with $\alpha = \beta \circ \gamma$, the ansatz P_2 is a transform of P_2' under γ, that is, $\gamma\colon P_2' \to P_2$. The expression (5.3.6) together with (5.2.2) leads to $(f',\ \psi') = (\rho\Delta_0,\ \Delta_1)$. This proves the corollary. $\square$

We notice that (5.3.12) is consistent with (5.3.1). We proceed to follow the same procedure as above in order to derive P_3 and P_3' such that

$$\beta\colon P_3 \to P_2', \qquad \alpha\colon P_3' \to P_2'.$$

Define P_2 by

$$\partial_\rho \Delta_2 = -\rho \partial_z \Delta_1, \qquad \partial_z \Delta_2 = \rho \partial_\rho \Delta_1 - 2\Delta_1. \tag{5.3.13}$$

Then we have for P_2

$$(\partial_\rho{}^2 - 3\rho^{-1}\partial_\rho + \partial_z{}^2)\Delta_2 = 0. \tag{5.3.14}$$

The expressions of P_3 and P_3' are given as follows.

Proposition 5.3.5. *By using* Δ_0, Δ_1 *and* Δ_2 *the ansatze* P_3 *and* P_3' *turn out to be*

$$P_3;(f,\psi) = \left(\frac{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}}{\Delta_0}, \frac{i\begin{vmatrix} \Delta_1 & \Delta_2 \\ -\Delta_0 & \Delta_1 \end{vmatrix}}{\Delta_0} \right), \tag{5.3.15}$$

$$P_3';(f,\psi) = \left(\frac{\begin{vmatrix} \rho\Delta_0 & \Delta_1 \\ -\Delta_1 & \rho\Delta_0 \end{vmatrix}}{\begin{vmatrix} \rho^2\Delta_0 & \Delta_1 & -\Delta_2 \\ -\Delta_1 & \Delta_0 & \Delta_1 \\ -\Delta_2 & -\Delta_1 & \rho^2\Delta_0 \end{vmatrix}}, \frac{i\begin{vmatrix} \Delta_1 & \Delta_2 \\ -\Delta_0 & \Delta_1 \end{vmatrix}}{\begin{vmatrix} \rho^2\Delta_0 & \Delta_1 & -\Delta_2 \\ -\Delta_1 & \Delta_0 & \Delta_1 \\ -\Delta_2 & -\Delta_1 & \rho^2\Delta_0 \end{vmatrix}} \right). \tag{5.3.16}$$

The proof is carried out in the same way as in Proposition 5.3.3 and Corollary 5.3.4, though rather complicated.

Proposition 5.3.5 suggests that we may express the ansatze P_l and P_l', $l \geq 4$, using determinants with entries composed of ρ and Δ_r, $r \geq 0$. The variables Δ_r are defined recursively by

$$\partial_\rho \Delta_r = -\rho \partial_z \Delta_{r-1},$$

$$\partial_z \Delta_r = \rho \partial_\rho \Delta_{r-1} + 2(1-r)\Delta_{r-1}. \tag{5.3.17}$$

Then each Δ_r should satisfy a linear equation

$$\{\partial_\rho{}^2 + (1-2r)\rho^{-1}\partial_\rho + \partial_z{}^2\}\Delta_r = 0. \tag{5.3.18}$$

Thus we have found out the hierarchy of P_l and P'_l, which is shown in the following diagram:

$$\begin{tikzcd}
& P_2 \arrow[r, leftrightarrow, "\alpha"] \arrow[d, leftrightarrow, "\gamma"] \arrow[dl, leftrightarrow, "\beta"'] & P_3 \arrow[d, leftrightarrow, "\gamma"] \arrow[dl, leftrightarrow, "\beta"'] & \arrow[l] \\
P'_1 \arrow[r, leftrightarrow, "\alpha"] & P'_2 \arrow[r, leftrightarrow, "\alpha"] & P'_3 & \arrow[l]
\end{tikzcd}$$

Here the bi-directed arrows $\xleftrightarrow{\beta}$ and $\xleftrightarrow{\gamma}$ indicate that β and γ are involutions.

Finally, we notice that P_l and P'_l (l: even) are real functions of ρ and z. On the other hand, P_l and P'_l (l: odd) give real metric coefficients (f, ω) by way of the mapping I.

5.4. Concrete solutions

In the previous section, we have presented a new method of constructing a family of exact solutions to the stationary axially symmetric Einstein equations. The metric coefficients are expressed by the ratio of determinants with entries characterized by the solutions of linear differential equations. In this section, we solve the linear equations with the recursion relations, explicitely. Then we give a new series of concrete solutions corresponding to the ansatze P_l and P'_l.

For the simplest ansatz P'_1, we use Lemma 5.3.2 to derive a solution (f, ψ). A special solution

$$\Delta_0 = R^{-1}, \qquad R = \{\rho^2 + (z-a)^2\}^{1/2}, \qquad (5.4.1)$$

where a is a real constant, of the cylindrical Laplace equation (5.3.5) gives a solution of (3.1.8) and (3.1.9) as

$$f = R, \quad \psi = -iR. \qquad (5.4.2)$$

By the mapping I, the solution (5.4.2) yields real metric coefficients

$$f = \rho R^{-1}, \quad \omega = R.$$

We can easily determine the metric coefficient Γ appearing in (3.1.1) by quadrature of (3.1.2). Then we have a metric

$$- ds^2 = \rho^{-1/2}(d\rho^2 + dz^2) - \rho R^{-1} dt^2 + 2\rho dt\, d\phi, \quad (5.4.3)$$

which includes one parameter.

Next, we consider the ansatze P_3 and P_3'. From (5.4.1), the recursion relations (5.3.7) and (5.3.17) are integrated to give the variables Δ_1 and Δ_2,

$$\Delta_1 = \{bR - (z - a)\}R^{-1}, \quad (5.4.4)$$

$$\Delta_2 = \{R^2 - 2b(z - c)R + (z - a)^2\}R^{-1}, \quad (5.4.5)$$

where a, b, c $\mathbb{R}$. It can be proved that Δ_1 and Δ_2 satisfy the linear equations (5.3.11) and (5.3.14), respectively. The metric coefficients corresponding to the ansatz P_3 and P_3' are described in terms of the variables Δ_0, Δ_1 and Δ_2. For example, the ansatz P_3 gives rise to real metric coefficients

$$f = \rho\{(b^2 + 1)R - 2b(z - a)\}^{-1},$$

$$\omega = \{(b^2 + 1)\rho^2 + (b^2 - 1)(Z - a)^2$$

$$+ 2b(2z - a - c)R\}R^{-1}, \quad (5.4.6)$$

which include three parameters.

Remark 5.4.1. As the ansatze P_2 and P_2' are real-valued, the metric coefficients (f, ω) are complex-valued.

To obtain real-valued ones (f, ω) from P_2 or P_2' we have to apply the definition (3.1.7). Since this application is equivalent to the transformation β, the resulting real metric coefficients coincide with those obtained through β from P_1' and P_3, respectively.

Finally, we set

$$\Delta_0 = \exp(\pm kz)\chi(\rho) \tag{5.4.7}$$

in the linear equation (5.3.5). Then we can derive for the ordinary differential equation

$$\frac{d^2}{d\rho^2}\chi(\rho) + \rho^{-1}\frac{d}{d\rho}\chi(\rho) + k^2\chi(\rho) = 0. \tag{5.4.8}$$

Therefore, it is known that a general solution of Equation (5.4.8) is a linear combination of the Bessel function $J_0(k\rho)$ and the Neumann function $N_0(k\rho)$. If we start with Δ_0 given by (5.4.7), we would obtain other series of metric coefficients (f, ω).

CHAPTER VI

BÄCKLUND TRANSFORMATIONS

FOR THE SELF-DUAL EQUATIONS

Gauge theory is now accepted as a governing principle in elementary particle physics, which was originated by Yang and Mills [13] and has been called steady attention of physicists and mathematicians. One of marked results in the gauge theory is the discovery of instanton solutions. These are finite action solutions of the four-dimensional Euclidean Yang-Mills equations and can be regarded as solitons in gauge fields. Recently, the soliton theory has been applied to the Yang-Mills equations. Several Bäcklund transformations which relate one solution to another are proposed in [5, 6, 8, 10]. However, as they do not generate solutions actually, these approaches are unsatisfactory for developing the transformation theory of gauge fields.

The purpose of this chapter is to present three types of Bäcklund transformations for the self-dual Yang-Mills equations by means of the Riemann-Hilbert problem. In Section 6.1, we review the useful formulation called the R gauge. The field equations which we discuss take the form of four-dimensional chiral field equations. A system of Zakharov-Shabat linear equations is derived by the method of prolongation in Section 6.2. We establish a transformation theorem by using the Riemann-Hilbert problem. In Sections 6.3, 6.4 and 6.5, three types of Bäcklund transformations are considered as applications of the Riemann-

Hilbert transformations.* The first transformation to be proposed in Section 6.3 gives N-instanton solutions via not analytic but algebraic calculations. The second to be presented in Section 6.4 relates the Atiyah-Ward ansatze to one another. The last transformation to be discussed in Section 6.5 keeps the reality conditions of $SU(n)$ gauge potentials.

6.1. Self-dual Yang-Mills gauge field equations

The dynamical variable in the Yang-Mills theory is a gauge potential

$$B = B_\mu(x)dx^\mu, \tag{6.1.1}$$

where the components $B_\mu(x)$ take their values in the Lie algebra g of a Lie group G called gauge group and $\mu = 1, 2, 3, 4$. The gauge field strength F is defined by

$$F = dB + B \wedge B = \frac{1}{2} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu \tag{6.1.2}$$

with components

---

* Sections 6.3, 6.4 and 6.5 are mainly the results of the collaborations with Lecturer Kimio UENO, Department of Mathematics, Yokohama City University.

$$F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) + [B_\mu(x), B_\nu(x)].$$

In this chapter, we work on real and complexified flat Euclidean spaces. Hence raising or lowing indices has no significance. The Euclidean Yang-Mills Lagrangian density [13] is

$$L = -\frac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}). \tag{6.1.3}$$

The gauge transformation is defined as a smooth map $g: \mathbb{R}^4 \to G$ and acts on the potentials and the field strengths as follows:

$$\begin{aligned} B_\mu \to B'_\mu &= gB_\mu g^{-1} - \partial_\mu g \cdot g^{-1}, \\ F_{\mu\nu} \to F'_{\mu\nu} &= gF_{\mu\nu}g^{-1}. \end{aligned} \tag{6.1.4}$$

Then it is clear that the Yang-Mills action

$$S[B] = -\frac{1}{2}\int_{\mathbb{R}^4} dx\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})$$

is invariant under the smooth gauge transformation.

We now state the Eular-Lagrangian equations for (6.1.3). Let us introduce covariant derivation by

$$D_\mu = \partial_\mu + [B_\mu, \quad]. \tag{6.1.5}$$

The extreme of the action $S[B]$ is found by usual calculus of variation techniques to be

$$D_\mu F_{\mu\nu} = 0. \tag{6.1.6}$$

These second order nonlinear partial differential equations for the gauge potentials B_μ are called *the Yang-Mills equations*. On the other hand, from the definition (6.1.2), it follows that

$$D_\mu {*F_{\mu\nu}} = 0, \tag{6.1.7}$$

where the symbol $*$ stands for Hodge's star operator in $\mathbb{R}^4$. Then $*F_{\mu\nu}$ is written as

$$*F_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma},$$

where $\varepsilon_{\mu\nu\rho\sigma}$ are the completely antisymmetric tensors with $\varepsilon_{1234} = 1$. The identity (6.1.7) is known as *the Bianchi identity* in gauge theory. If the gauge group G is $U(1)$, then the Yang-Mills equations (6.1.6) and the identity (6.1.7) are reduced to well-known Maxwell's electromagnetic field equations in vacuua.

Comparing (6.1.7) to the Yang-Mills equations (6.1.6) we see that any field strengths which are self-dual

$$*F_{\mu\nu} = F_{\mu\nu} \tag{6.1.8}$$

automatically satisfy the Yang-Mills equations. These first order nonlinear equations are referred to as *the self-dual (Yang-Mills) equations*. If some solutions B_μ

of (6.1.8) are anti-hermitian and trace-free n×n matrices (that is, they belong to the Lie algebra *su*(n)), they are called *su*(n) *real* gauge potentials.

In the subsequent discussion, we should rather treat B_μ to be analytically continued into the *complex* space $\mathbb{C}^4$ where x^μ are complex. Then B_μ are complex gauge potentials defined in $\mathbb{C}^4$. This idea was first proposed by Yang [12]. Let us introduce complex variables y, $\bar{y}$, z, $\bar{z}$ through

$$y = \tfrac{1}{2}(x^1 + ix^2), \quad \bar{y} = \tfrac{1}{2}(x^1 - ix^2),$$
$$z = \tfrac{1}{2}(x^3 - ix^4), \quad \bar{z} = \tfrac{1}{2}(x^3 + ix^4). \tag{6.1.9}$$

Then the self-dual equations (6.1.8) take the form

$$F_{yz} = 0, \quad F_{\bar{y}\bar{z}} = 0, \tag{6.1.10}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0, \tag{6.1.11}$$

where complex gauge strengths are defined by

$$F_{yz} = \partial_y B_z - \partial_z B_y + [B_y, B_z],$$

$$B_y = B_1 - iB_2, \quad B_z = B_3 + iB_4,$$

ans so on. The equations (6.1.10) can be easily integrated as follows:

$$B_y = D^{-1}\partial_y D, \quad B_{\bar{y}} = \bar{D}^{-1}\partial_{\bar{y}}\bar{D},$$

$$B_z = D^{-1}\partial_z D, \quad B_{\bar{z}} = \bar{D}^{-1}\partial_{\bar{z}}\bar{D}. \qquad (6.1.12)$$

Here D and $\bar{D}$ are $GL(n, \mathbb{C})$ arbitrary matrix functions of $(y, \bar{y}, z, \bar{z})$. An important observation was made by [8, 10]. Let us define a $GL(n, \mathbb{C})$ matrix P by

$$P = D\bar{D}^{-1}. \qquad (6.1.13)$$

The ramaining field equations (6.1.11) lead to

$$\partial_y(\partial_{\bar{y}}P \cdot P^{-1}) + \partial_z(\partial_{\bar{z}}P \cdot P^{-1}) = 0. \qquad (6.1.14)$$

For a solution P of (6.1.14), we may reconstruct gauge potentials B_μ through (6.1.12) and (6.1.13). Thus we are left with Equations (6.1.14), a reduced form of the original self-dual equations (6.1.8). We simply call (6.1.14) *the* ($GL(n, \mathbb{C})$) *self-dual equations*. If a solution P is a positive definite hermitian matrix whose determinant is equal to 1, the resulting gauge potentials are $su(n)$ real. Conversely, if B_μ are $su(n)$ real gauge potentials, we can take $\bar{D}$ to be $(D^\dagger)^{-1}$ for real x^μ, where $\dagger$ indicates the hermitian conjugate. Therefore, P is a positive unimodular hermitian matrix. We remark here that we can obtain an $SL(n, \mathbb{C})$ solution from $GL(n, \mathbb{C})$ solution according to the following lemma.

Lemma 6.1.1. *Suppose* P *be a* $GL(n, \mathbb{C})$ *solution of* (6.1.14) *Then* P' *defined by*

$$P' = (\det P)^{-1/n} P \tag{6.1.15}$$

is an $SL(n, \mathbb{C})$ *solution.*

Proof. It is obvious that $\det P' = 1$. Using the identity

$$\partial_\mu(\det P) = \det P \cdot \mathrm{Tr}(\partial_\mu P \cdot P^{-1}),$$

we can derive

$$\partial_y(\partial_{\bar{y}} P' \cdot P'^{-1}) + \partial_z(\partial_{\bar{z}} P' \cdot P'^{-1}) = 0.$$

This proves the lemma. □

Yang [12] proposed gauge fixing for $SL(2, \mathbb{C})$ solution P such that D and $\bar{D}$ are $SL(2, \mathbb{C})$ triangular matrices with $D^\dagger = \bar{D}^{-1}$. This gauge called *the* R *gauge*, a fixed choice of the form of D and $\bar{D}$, takes the form

$$P = \phi^{-1} \begin{pmatrix} 1 & \bar{\rho} \\ \rho & \phi^2 + \rho\bar{\rho} \end{pmatrix}, \quad D = \phi^{-1/2} \begin{pmatrix} 1 & 0 \\ \rho & \phi \end{pmatrix}, \tag{6.1.16}$$

where ϕ is real and $\rho = \bar{\rho}^*$ for real x^μ. Then the real gauge potentials are given by

$$B_y = \phi^{-1}\begin{pmatrix} -\phi_y & 0 \\ 2\rho_y & \phi_y \end{pmatrix}, \quad B_{\bar{y}} = \phi^{-1}\begin{pmatrix} \phi_{\bar{y}} & -2\bar{\rho}_{\bar{y}} \\ 0 & -\phi_{\bar{y}} \end{pmatrix},$$

$$B_z = \phi^{-1}\begin{pmatrix} -\phi_z & 0 \\ 2\rho_z & \phi_z \end{pmatrix}, \quad B_{\bar{z}} = \phi^{-1}\begin{pmatrix} \phi_{\bar{z}} & -2\bar{\rho}_{\bar{z}} \\ 0 & -\phi_{\bar{z}} \end{pmatrix}, \quad (6.1.17)$$

where $\phi_y = \partial_y\phi$ and so on.

Incidentally, we may factrize P as

$$P = f^{-1}\begin{pmatrix} 1 & g \\ e & f^2 + eg \end{pmatrix}. \qquad (6.1.18)$$

substituting (6.1.18) into (6.1.14) we obtain

$$f\,\Box f - f_y f_{\bar{y}} - f_z f_{\bar{z}} + e_y g_{\bar{y}} + e_z g_{\bar{z}} = 0,$$

$$f\,\Box e - 2e_y f_{\bar{y}} - 2e_z f_{\bar{z}} = 0,$$

$$f\,\Box g - 2g_{\bar{y}} f_y - 2g_{\bar{z}} f_z = 0, \qquad (6.1.19)$$

where $\Box$ is a four-dimensional Laplacian: $\Box = \partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}}$.

The gauge potentials are given by

$$B_y = f^{-1}\begin{pmatrix} f_y & 0 \\ 2g_{\bar{z}} & -f_y \end{pmatrix}, \quad B_{\bar{y}} = f^{-1}\begin{pmatrix} -f_{\bar{y}} & 2e_z \\ 0 & f_{\bar{y}} \end{pmatrix},$$

$$B_z = f^{-1}\begin{pmatrix} f_z & 0 \\ -2g_{\bar{y}} & -f_z \end{pmatrix}, \quad B_{\bar{z}} = f^{-1}\begin{pmatrix} -f_{\bar{z}} & -2e_y \\ 0 & f_z \end{pmatrix}. \qquad (6.1.20)$$

The proof of (6.1.20) is straightforward through the decomposition $P = D\bar{D}^{-1}$ along with the use of (6.1.19). Equations (6.1.19) are called *the Yang equations.*

Remark 6.1.2. The gauge potentials (6.1.17) and (6.1.20) are related by a set of transformations

$$\phi = f^{-1},$$

$$\phi^{-1}\rho_y = f^{-1}g_{\bar{z}}, \quad \phi^{-1}\rho_z = -f^{-1}g_{\bar{y}},$$

$$\phi^{-1}\bar{\rho}_{\bar{y}} = -f^{-1}e_z, \quad \phi^{-1}\bar{\rho}_{\bar{z}} = f^{-1}e_y. \qquad (6.1.21)$$

In the subsequent sections, we will consider three-types of Bäcklund transformations to construct $GL(n, \mathbb{C})$ solutions of the self-dual equations.

6.2. Riemann-Hilbert transformation

In Section 4.3, we have discussed the Riemann-Hilbert transformation for three-dimensional Einstein equations.

Along this line, we prove a transformation theorem for the self-dual equations. First we derive a Zakharov-Shabat system of inverse scattering formula by the method of prolongation developed in Section 2.3. The self-dual equations to be considered can be written (see (6.1.14)) as

$$\partial_z A_1 + \partial_y A_2 = 0,$$

$$A_1 = \partial_{\bar{z}} P \cdot P^{-1}, \qquad A_2 = \partial_{\bar{y}} P \cdot P^{-1}. \tag{6.2.1}$$

Then the identity (zero curvature conditions)

$$\partial_{\bar{y}} A_1 - \partial_{\bar{z}} A_2 + [A_1, A_2] = 0 \tag{6.2.2}$$

should hold. Let us define a set of 4-forms

$$\alpha_1 = (dA_1 \wedge dy - dA_2 \wedge dz) \wedge d\bar{y} \wedge d\bar{z},$$

$$\alpha_2 = (dA_1 \wedge d\bar{z} + dA_2 \wedge d\bar{y} + [A_1, A_2] d\bar{y} \wedge d\bar{z}) \wedge dy \wedge dz. \tag{6.2.3}$$

Equations (6.2.1) and (6.2.2) can be expressed in terms of the closed ideal spanned by α_1 and α_2. To prolong the ideal we define a 3-form by

$$\begin{aligned}\Theta = &- dY \wedge \{\zeta^{-1} d\bar{y} \wedge dz + (dy \wedge d\bar{y} + dz \wedge d\bar{z}) + \zeta d\bar{y} \wedge d\bar{z}\} \\ &+ A_1 Y\, dy \wedge d\bar{z} \wedge (\zeta^{-1} dz + d\bar{y}) \\ &- A_2 Y\, d\bar{y} \wedge dz \wedge (\zeta^{-1} dy - d\bar{z}).\end{aligned} \tag{6.2.4}$$

Here $Y = Y(y, \bar{y}, z, \bar{z};\zeta)$ is an $n\times n$ matrix and ζ is a complex parameter. By sectioning Θ onto the solutions of (6.2.1) and (6.2.2), we have the following proposition.

Proposition 6.2.1 *Let* D_1 *and* D_2 *be linear differential operators:*

$$D_1 = \zeta^{-1}\partial_{\bar{z}} + \partial_y, \quad D_2 = \zeta^{-1}\partial_{\bar{y}} - \partial_z. \tag{6.2.5}$$

Then the compatibility condition of the linear differential equations

$$D_k Y(\zeta) = \zeta^{-1} A_k Y(\zeta), \quad k = 1, 2, \tag{6.2.6}$$

is equivalent to the self-dual equations (6.1.14).

Furthermore, if $Y(\zeta)$ *is a fundamental solution matrix of* (6.2.6), *analytic near* $\zeta = 0$, *then a solution* P *of* (6.1.14) *is given by*

$$P = Y(0). \tag{6.2.7}$$

Proof. Since $\Theta = 0$ on the solutions of (6.2.1) and (6.2.2), we see that

$$(D_1 Y - \zeta^{-1} A_1 Y)(\zeta d\bar{y} - dz)\wedge dy\wedge d\bar{z}$$

$$= (D_2 Y - \zeta^{-1} A_2 Y)(\zeta d\bar{z} - dy)\wedge d\bar{y}\wedge dz.$$

This proves the compatibility of (6.2.6). For the remaining part of the theorem, by setting $\zeta = 0$ in (6.2.6) we immediately obtain (6.2.7). This completes the proof. □

Equations (6.2.6) gives a Zakharov-Shabat system for the self-dual equations (see Chapter III). Proposition 6.2.1 shows us that a transformation on fundamental solution matrices of (6.2.6) induces a transformation on solutions of (6.1.14). This consideration allows us to set up a transformation theory by making use of the Riemann-Hilbert problem for (6.2.6).

We now state the Riemann-Hilbert transformation for the self-dual equations. Let C be a closed analytic curve in the complex ζ-place encircling the origin so that a matrix $Y(\zeta)$ is analytic in $C \cup C_+$. Here C_+ denotes the inside of C, while the outside is written as C_-. Let us consider the Riemann-Hilbert problem of finding matrices $X_\pm(\zeta)$ which are analytic and invertible in $C \cup C_\pm$, respectively, and satisfy

$$X_-(\zeta) = X_+(\zeta)H(\zeta), \quad \zeta \in C, \tag{6.2.8}$$

and a normalization condition

$$X_-(\infty) = 1. \tag{6.2.9}$$

The n×n matrix $H(\zeta)$ is defined by

$$H(\zeta) = Y(\zeta)u(\zeta)Y(\zeta)^{-1} \qquad (6.2.10)$$

and $u(\zeta) = u(y, \bar{y}, z, \bar{z};\zeta)$ is a $GL(n, \mathbb{C})$ matrix which analytic in $\zeta \in C$ such that

$$D_k u(\zeta) = 0. \qquad (6.2.11)$$

We set variables w, w_1 and w_2:

$$w = \zeta^{-1}, \quad w_1 = \bar{z} - wy, \quad w_2 = \bar{y} + wz. \qquad (6.2.12)$$

The conditions (6.2.11) imply that $u(\zeta)$ is an arbitrary function of w, w_1 and w_2, three of five independent variables (see also (4.3.26)). It should be noted that

$$\Box\, u(\zeta) = 0.$$

We assume that there is a pair of fundamental solution matrices $X_\pm(\zeta)$ of the above Riemann-Hilbert problem. If $u(\zeta)$ is very close to unit matrix, this problem actually has the solution. Let us define the n×n matrix

$$Y'(\zeta) = \begin{cases} X_+(\zeta)Y(\zeta) & \text{in } C_+, \\ X_-(\zeta)Y(\zeta)u(\zeta)^{-1} & \text{in } C_-, \end{cases} \qquad (6.2.13)$$

$$A_1' = A_1 + \partial_y(\partial_w X_-|_{w=0}),$$

$$A_2' = A_2 - \partial_z(\partial_w X_-|_{w=0}). \tag{6.2.14}$$

Then we have the following main theorem.

Theorem 6.2.2. *The matrix* $Y'(\zeta)$ *is a fundamental solution matrix of*

$$D_k Y'(\zeta) = \zeta^{-1} A_k' Y'(\zeta), \quad k = 1, 2. \tag{6.2.15}$$

The n×n *matrix* P' *defined by* $P' = Y'(0)$ *is a solution of the self-dual equations.*

Proof. We follow the line of the proof of Theorem 4.3.4 in principle. Since $u(\zeta)$ is annihilated by the operators D_k, we obtain on C

$$D_k X_+ \cdot X_+{}^{-1} + \zeta^{-1} X_+ A_k X_+{}^{-1} = D_k X_- \cdot X_-{}^{-1} + \zeta^{-1} X_- A_k X_-{}^{-1}$$

from (6.2.6) and (6.2.13). We write $X(\zeta) = X_+(\zeta)$, (resp. $X_-(\zeta)$), in C_+, (resp. C_-). These equations imply that

$$D_k X(\zeta) \cdot X(\zeta)^{-1} + \zeta^{-1} X(\zeta) A_k X(\zeta)^{-1}$$

are rational functions of ζ, and have a simple zero at infinity. Using the Laurent expansions at infinity together with (6.2.14), we have, after a calculation,

$$D_k X(\zeta) + \zeta^{-1} X(\zeta) A_k = \zeta^{-1} A_k' X(\zeta).$$

We multiply these by $Y(\zeta)$ from the right, then we derive (6.2.15) in C_+. Since the compatibility condition of (6.2.15) is also the self-dual equations, P' is a solution. This completes the proof. $\square$

The construction of the self-dual solutions through the linear differential equations (6.2.6) is carried out by the above theorem. We remark that if P and $u(\zeta)$ are $SL(n, \mathbb{C})$-valued, the resulting P' is also an $SL(n, \mathbb{C})$ solution of the self-dual equations.

6.3. Construction of instanton solutions

The Riemann-Hilbert transformation (6.2.13) is completely characterized by $u(\zeta)$. In this section, we consider the Riemann-Hilbert transformation induced by

$$u(\zeta) = 1 + \sum_{j=1}^{N} \left(\frac{a'_j}{\zeta - \alpha_j} + \frac{b'_j}{\zeta - \beta_j} + \frac{c_j}{\zeta - \zeta_j}\right) K , \qquad (6.3.1)$$

where

$$a'_j = \frac{a_j}{\bar{z} - \bar{z}_j} , \quad b'_j = \frac{b_j}{\bar{y} - \bar{y}_j} , \quad \alpha_j = \frac{y - y_j}{\bar{z} - \bar{z}_j} , \quad \zeta_j = - \frac{z - z_j}{\bar{y} - \bar{y}_j} ,$$

and 1 is the n×n unit matrix and K is an n×n constant matrix such that $K^2 = 0$. Here a_j, b_j, c_j, y_j, $\overline{y}_j$, z_j, $\overline{z}_j$ and ζ_j are complex constants. Since $K^2 = 0$ and $a_j'(\zeta - \alpha_j)^{-1} = a_j w(w_1 - \overline{z}_j + w y_j)^{-1}$ et al., it follows that $u(\zeta) \in SL(n, \mathbb{C})$ and $D_k u(\zeta) = 0$, $k = 1, 2$. We assume that α_j, β_j and ζ_j are mutually distinct and located in C_+. We have the following theorem.

Theorem 6.3.1. *Let* $Y(\zeta)$ *be a fundamental solution matrix of* (6.2.6). *Then a solution matrix* $X\_(\zeta)$ *of the Riemann-Hilbert problem with* (6.3.1) *can take the form*

$$X\_(\zeta) = 1 + \sum_{j=1}^{N} \left(\frac{R_j}{\zeta - \alpha_j} + \frac{S_j}{\zeta - \beta_j} + \frac{T_j}{\zeta - \zeta_j}\right), \qquad (6.3.2)$$

where R_j, S_j *and* T_j *are given by solving linear equations*

$$(R_1, \ldots, R_N, S_1, \ldots, S_N, T_1, \ldots, T_N)W$$
$$= (a_1'V(\alpha_1), \ldots, a_N'V(\alpha_N), b_1'V(\beta_1), \ldots$$
$$\ldots, b_N'V(\beta_N), c_1V(\zeta_1), \ldots, c_NV(\zeta_N)). \qquad (6.3.3)$$

Here $W = (W^{(kl)})_{1 \le k, l \le 3}$ *is a* 3nN×3nN *matrix such that each* (ij)*-minor block* $W_{ij}^{(kl)}$ *of* $W^{(kl)}$, $1 \le i,j \le N$, *is*

$$W_{ij}^{(11)} = \frac{a_j'}{\alpha_i - \alpha_j} V(\alpha_j), \ (i \neq j), \qquad W_{jj}^{(11)} = 1 - a_j' \dot{V}(\alpha_j),$$

$$W_{ij}^{(12)} = \frac{a_j'}{\beta_i - \alpha_j} V(\alpha_j), \qquad W_{ij}^{(13)} = \frac{a_j'}{\zeta_i - \alpha_j} V(\alpha_j),$$

$$W_{ij}^{(21)} = \frac{b_j'}{\alpha_i - \beta_j} V(\beta_j), \qquad W_{ij}^{(22)} = \frac{b_j'}{\beta_i - \beta_j} V(\beta_j), \ (i \neq j),$$

$$W_{jj}^{(22)} = 1 - b_j' \dot{V}(\beta_j), \qquad W_{ij}^{(23)} = \frac{b_j'}{\zeta_i - \beta_j} V(\zeta_j),$$

$$W_{ij}^{(31)} = \frac{c_j}{\alpha_i - \zeta_j} V(\zeta_j), \qquad W_{ij}^{(31)} = \frac{c_j}{\beta_i - \zeta_j} V(\zeta_j),$$

$$W_{ij}^{(33)} = \frac{c_j}{\zeta_i - \zeta_j} V(\zeta_j), \ (i \neq j), \qquad W_{ij}^{(33)} = 1 - c_j \dot{V}(\zeta_j).$$

The matrices V *and* $\dot{V}$ *are defined by*

$$V(\gamma_j) = Y(\gamma_j)\, K\, Y(\gamma_j)^{-1}, \qquad \dot{V}(\gamma_j) = \dot{Y}(\gamma_j)\, K\, Y(\gamma_j)^{-1},$$

with $\gamma_j = \alpha_j,\ \beta_j,\ \zeta_j$, *where* $\dot{Y}$ *stands for the derivative of* Y *with respect to* ζ.

Proof. Assume that $X_-(\zeta)$ takes the form of (6.3.2) with R_j, S_j and T_j undetermined. Since $X_+(\zeta) = X_-(\zeta)H(\zeta)$ must be analytic in C_+, the coefficients of

$(\zeta - \alpha_j)^{-2}$, $(\zeta - \beta_j)^{-2}$ and $(\zeta - \zeta_j)^{-2}$ vanish. Then we obtain

$$R_j V(\alpha_j) = 0, \quad S_j V(\beta_j) = 0, \quad T_j V(\zeta_j) = 0. \tag{6.3.4}$$

In addition, the residue of $X_+(\zeta)Y(\zeta)$ at $\zeta = \alpha_j$ must also vanish. Then we have

$$R_j(1 - a_j^{\prime}\dot{V}(\alpha_j)) + a_j \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{R_i}{\alpha_i - \alpha_j} V(\alpha_j)$$

$$+ a_j \sum_{i=1}^{N} \left(\frac{S_j}{\beta_i - \alpha_j} + \frac{T_j}{\zeta_i - \alpha_j}\right) V(\alpha_j) = a_j V(\alpha_j)^{\prime}. \tag{6.3.5}$$

Similar equations are true at $\zeta = \beta_j$, ζ_j. We observe that the first equation of (6.3.4) follows from (6.3.5) by multiplying (6.3.5) by $V(\alpha_j)$ from the right and by using $K^2 = 0$. The remainding equations in (6.3.4) follow from similar equations to (6.3.5) for $\zeta = \beta_j$ and ζ_j in the same way. Equation (6.3.5) can be written as part of (6.3.3). Thus the proof can be continued in this way. □

We obtain the solution $X_\pm(\zeta)$ of the Riemann-Hilbert problem associated with (6.3.1) by an elementary operation in linear algebra. Now we call (6.2.13) with (6.3.1) the algebraic Riemann-Hilbert transformation or the Bäcklund transformation.

As an application of Theorem 6.3.1, we construct 't Hooft's N-instanton solution of $SU(2)$ self-dual equations. Let us introduce 2×2 matrices $g^{(i)}(\zeta)$ by

$$g^{(i)}(\zeta) = \begin{pmatrix} \dfrac{1+q^{(i+1)}(\zeta)}{1+q^{(i)}(\zeta)} & \dfrac{q^{(i)}(\zeta)-q^{(i+1)}(\zeta)}{1+q^{(i)}(\zeta)} \displaystyle\sum_{j=1}^{N} \frac{a_j}{\zeta-\alpha_j} \\ 0 & 1 \end{pmatrix},$$

$1 \le i \le N-1$, where

$$q^{(0)}(\zeta) = 0, \quad q^{(i)}(\zeta) = \frac{\zeta}{\zeta(\bar{y}-\bar{y})_i + (z-z_j)},$$

$1 \le i \le N$. Obviously $D_k g^{(i)}(\zeta) = 0$, $k = 1, 2$ and $g^{(i)}(0) = 1$ for any i. We discuss the Bäcklund transformation associated with

$$u^{(j)}(\zeta) = 1 + \frac{a_j}{\zeta-\alpha_j} K, \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{6.3.6}$$

$1 \le j \le N$. For a trivial solution $Y'^{(0)}(\zeta) = 1$ of (6.2.6), we carry out successive transformations defined by

$$Y^{(j)}(\zeta) = Y'^{(j)}(\zeta) g^{(j)}(\zeta), \tag{6.3.7}$$

$$Y'^{(j+1)}(\zeta) = u^{(j+1)}(\zeta) \circ Y^{(j)}(\zeta), \tag{6.3.8}$$

$0 \le j \le N-1$. Here the solution $Y^{(j)}(\zeta)$ of (6.2.6) is given by multiplying $Y'^{(j)}(\zeta)$ by $g^{(j)}(\zeta)$. The symbol

$u^{(j+1)}(\zeta)\circ$ denotes the Bäcklund transformation induced by $u^{(j+1)}(\zeta)$. We have

Proposition 6.3.2. *The resulting fundamental solution matrix* $Y'^{(N)}(\zeta)$ *gives a solution*

$$P'^{(N)} = Y'^{(N)}(0)$$
$$= \begin{pmatrix} 1 & -\sum_{j=1}^{N} \dfrac{a_j}{(y-y_j)(\bar{y}-\bar{y}_j)+(z-z_j)(\bar{z}-\bar{z}_j)} \\ 0 & 1 \end{pmatrix}$$

of the self-dual equations (6.1.14).

Proof. The proof is given by induction. First we calculate $Y'^{(1)}(\zeta)$. From Theorem 6.3.1, the Bäcklund transformation induced by $u^{(1)}(\zeta)$ takes the form

$$Y'^{(1)}(\zeta) = X_-^{(1)}(\zeta)Y^{(0)}(\zeta)u^{(1)}(\zeta)^{-1}$$

where $X_-^{(1)}(\zeta) = 1 + R_1/(\zeta - \alpha_1)$ and

$$R_1 = a_1' \begin{pmatrix} 0 & 1+q^{(1)}(\alpha_1) \\ 0 & 0 \end{pmatrix}.$$

Hence we obtain

$$Y'^{(1)}(\zeta) = \begin{pmatrix} 1+q^{(1)}(\zeta) & \dfrac{a_1'}{\zeta-\alpha_1}(q^{(1)}(\alpha_1) - q^{(1)}(\zeta)) \\ 0 & 1 \end{pmatrix}.$$

Next we show that

$$Y'^{(k)}(\zeta) = \begin{pmatrix} 1+q^{(k)}(\zeta) & \sum_{j=1}^{k} \frac{a'_j}{\zeta-\alpha_j}(q^{(j)}(\alpha_j)-q^{(k)}(\zeta)) \\ 0 & 1 \end{pmatrix}, \quad (6.3.9)$$

by the induction with respect to k. Suppose that we have proved the k-th induction step. From Theorem 6.3.1 and (6.3.7), the Bäcklund transformation induced by $u^{(k+1)}(\zeta)$ is

$$Y'^{(k+1)}(\zeta) = X_-^{(k+1)}(\zeta)Y^{(k)}(\zeta)u^{(k+1)}(\zeta)^{-1},$$

$$X_-^{(k+1)}(\zeta) = 1 + R_{k+1}/(\zeta-\alpha_{k+1}),$$

$$R_{k+1} = a'_{k+1}\begin{pmatrix} 0 & 1+q^{(k+1)}(\alpha_{k+1}) \\ 0 & 0 \end{pmatrix}.$$

Then we obtain

$$Y'^{(k+1)}(\zeta) = \begin{pmatrix} 1+q^{(k+1)}(\zeta) & \sum_{j=1}^{k+1} \frac{a'_j}{\zeta-\alpha_j}(q^{(j)}(\alpha_j)-q^{(k+1)}(\zeta)) \\ 0 & 1 \end{pmatrix},$$

This proves (6.3.9) for any k, $1 \le k \le N$. We substitute $\zeta = 0$ into (6.3.9) with $k = N$ to obtain from Proposition 6.2.1 the solution $P'^{(N)}$. $\square$

By a simple $SL(2, \mathbb{C})$ transformation

$$P^{(N)} = P'^{(N)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have a solution of the $SL(2, \mathbb{C})$ self-dual equations. In the factrization (6.1.18), $P^{(N)}$ is expressed as

$$\begin{aligned} e &= f = -g \\ &= \{1 + \sum_{j=1}^{N} \frac{a_j}{(y - y_j)(\bar{y} - \bar{y}_j) + (z - z_j)(\bar{z} - \bar{z}_j)}\}^{-1}. \end{aligned} \tag{6.3.10}$$

If a_j are real and $y_j = \bar{y}_j^*$, $z_j = \bar{z}_j^*$, $1 \le j \le N$, f is real and $e = -g^*$ for real x^μ. Then P is $SU(1, 1)$-valued, so that the corresponding gauge fields have $SU(1, 1)$ gauge symmetry from the definition (6.1.13). The gauge potentials are not real. However, the transformations (6.1.21) give from $e = f = -g$ the relation, by setting $\phi = f^{-1}$,

$$\rho_y = \phi_{\bar{z}}, \quad \rho_z = -\phi_{\bar{y}}, \quad \bar{\rho}_{\bar{y}} = \phi_z, \quad \bar{\rho}_{\bar{z}} = -\phi_y. \tag{6.3.11}$$

The ansatz ϕ, ρ and $\bar{\rho}$ satisfying (6.3.11) implies that $\rho = \bar{\rho}^*$ and is called *the 't Hooft ansatz* of $SU(2)$ gauge potentials (see [9]). Providing that $a_j \in \mathbb{R}$, $y_j = \bar{y}_j^*$, $z_j = \bar{z}_j^*$, we obtain *the 't Hooft N-instanton solution*

$$\phi = 1 + \sum_{j=1}^{N} \frac{a_j}{(y - y_j)(\bar{y} - \bar{y}_j) + (z - z_j)(\bar{z} - \bar{z}_j)}, \tag{6.3.12}$$

which satisfies the four-dimensional Laplace equation $\Box\phi = 0$. It should be noted that the 't Hooft ansatz reduces the Lagrangian density (6.1.3) to

$$L = -\frac{1}{2}\Box\Box \ln \phi. \qquad (6.3.13)$$

We insert (6.3.12) into (6.3.13) and integrate over four-dimensional Euclidean space. By excluding the singularities in (6.3.12) from the region of integration and using Gauss' divergence theorem, we can show that

$$S[B] = 8\pi^2 N, \quad N = 1, 2, \ldots . \qquad (6.3.14)$$

Thus the instanton solution (6.3.12) gives a finite action. This solution has 5N arbitrary parameters. We can interprete them as the positions and scales of each single instanton. We remark that N-instanton solution which leads to (6.3.14) can include $8N - 3$ arbitrary parameters. This complete instanton solution is realized in the Atiyah-Drinfeld-Hitchin-Manin construction [1].

6.4. Atiyah-Ward ansatze

As an application of the Riemann-Hilbert transformation (6.2.13), we consider the second Bäcklund transformation which relates the Atiyah-Ward ansatze to one another. We start with a review of the Atiyah-Ward ansatze. For generating instanton solutions of the $SU(2)$ self-dual equations, Ward [11] and Atiyah-Ward [2] proposed an analytic vector bundle of rank 2 on $\mathbb{P}^3$, the three-dimensional complex projective space, with a transition matrix

$$T(y, y, z, z; \zeta) = \begin{pmatrix} \zeta^{-N} & \psi(\zeta) \\ 0 & \zeta^{N} \end{pmatrix}. \tag{6.4.1}$$

Here N is an integer not less than 1, and $\psi(\zeta) = \psi(y, y, z, z; \zeta)$ satisfies

$$D_k\psi(\zeta) = 0, \qquad k = 1, 2. \tag{6.4.2}$$

Decomposing $T(\zeta)$ into

$$T(\zeta) = Y^{(+)}(\zeta)^{-1}\, Y^{(-)}(\zeta), \tag{6.4.3}$$

where $Y^{(+)}(\zeta)$ and $Y^{(-)}(\zeta)$ are matrix functions analytic away from $\zeta = \infty$ and $\zeta = 0$, respectively, one obtains

$$\begin{aligned} D_1 Y^{(\pm)}(\zeta)\, Y^{(\pm)}(\zeta)^{-1} &= -(B_y + \zeta^{-1}B_{\bar{z}}), \\ D_2 Y^{(\pm)}(\zeta)\, Y^{(\pm)}(\zeta)^{-1} &= (B_z - \zeta^{-1}B_{\bar{y}}), \end{aligned} \tag{6.4.4}$$

where B's are independent of ζ. The compatibility condition of (6.4.4) for $Y^{(\pm)}(\zeta)$ is equivalent to the self-dual equations (6.1.10) and (6.1.11). Consequently, B's become self-dual potentials. Atiyah and Ward claimed the decomposition (6.4.3) yields N-instanton solution. The self-dual solutions B's and hence P given by the transition matrix $T(\zeta)$ with N are referred to as *the Atiyah-Ward ansatz* A_N. The ansatz A_1 gives the 't Hooft ansatz which satisfies (6.3.11).

Remark 6.4.1. The fundamental solution matrix $Y(\zeta)$ of (6.2.6) is related to $Y^{(\pm)}(\zeta)$ by the formula

$$Y(\zeta) = Y^{(-)}(\infty)^{-1} Y^{(+)}(\zeta). \tag{6.4.5}$$

Then the solution P_N of the self-dual equations corresponding to the ansatz A_N is given by

$$P_N = Y^{(-)}(\infty)^{-1} Y^{(+)}(0). \tag{6.4.6}$$

Making this approach further, Corrigan, Fairlie, Yates and Goddard [6] succeeded in giving an explicit expression to the self-dual solutions belonging to A_N by using the expansion

$$\psi(\zeta) = \sum_{n=-\infty}^{\infty} \Delta_n \zeta^n. \tag{6.4.7}$$

Since the conditions (6.4.2) yield

$$\partial_{\bar{z}} \Delta_n = -\partial_y \Delta_{n-1}, \qquad \partial_{\bar{y}} \Delta_n = \partial_z \Delta_{n-1}, \tag{6.4.8}$$

each Δ_n satisfies the Laplace equation $\Box \Delta_n = 0$. In our notation, their results are stated as follows: Let $P_N = (p_{N,ij})$, $i, j = 1, 2$ and let $D_n^{(m)}$ be

$$D_n^{(m)} = \begin{vmatrix} \Delta_{m-n} & \Delta_{m-n+1} & \cdots & \Delta_m \\ \Delta_{m-n+1} & \Delta_{m-n+2} & \cdots & \Delta_{m+1} \\ \vdots & \vdots & & \vdots \\ \Delta_m & \Delta_{m+1} & \cdots & \Delta_{m+n} \end{vmatrix}. \tag{6.4.9}$$

Then each element of P_N takes the form

$$p_{N,11} = (-1)^N D_{N-1}^{(-1)} / D_{N-1}^{(0)},$$

$$p_{N,21} = (-1)^{N+1} D_{N-2}^{(0)} / D_{N-1}^{(0)},$$

$$p_{N,12} = (-1)^{N+1} D_N^{(0)} / D_{N-1}^{(0)},$$

$$p_{N,22} = (-1)^N D_{N-1}^{(1)} / D_{N-1}^{(0)}. \tag{6.4.10}$$

Corrigan et al. derived the solution P_N in the factrization (6.1.18) by applying their Bäcklund transformations.

Let us define $p_{N,ij}^{(m)}$ by

$$p_{N,11}^{(m)} = (-1)^N D_{N-1}^{(m-1)} / D_{N-1}^{(m)},$$

$$p_{N,21}^{(m)} = (-1)^{N+1} D_{N-2}^{(m)} / D_{N-1}^{(m)},$$

$$p_{N,12}^{(m)} = (-1)^{N+1} D_N^{(m)} \;/\; D_{N-1}^{(m)},$$

$$p_{N,22}^{(m)} = (-1)^{N} D_N^{(m+1)} \;/\; D_{N-1}^{(m)}, \tag{6.4.11}$$

Since the superscripts of $P_N^{(m)}$ indicate that the entries of $D_n^{(m)}$ are shifted by m, $P_N^{(m)} = (p_{N,ij}^{(m)})$ is a solution of the self-dual equations (6.1.14) and especially $P_N^{(0)} = P_N$. It is said that $P_N^{(m)}$ belongs to a generalized Atiyah-Ward ansatz $A_N^{(m)}$. In the following discussion, we apply the Riemann-Hilbert transformation to derive $P_N^{(m)}$ from the t' Hooft ansatz $P_1^{(0)}$.

Let $Y(\zeta)$ be a fundamental solution matrix of (6.2.6) analytic near $\zeta = 0$. We consider the Riemann-Hilbert transformation induced by

$$u(\zeta) = \zeta E_1 + \zeta^{-1} E_2, \tag{6.4.12}$$

$$\text{with} \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $Y'(\zeta)$ be the resulting fundamental solution matrix

$$Y'(\zeta) = X_-(\zeta) Y(\zeta) u(\zeta)^{-1}. \tag{6.4.13}$$

Denote Y_n and Y'_n be the coefficients of the Taylor exapnsion of $Y(\zeta)$ and $Y'(\zeta)$, respectively,

$$Y(\zeta) = \sum_{n=0}^{\infty} Y_n \zeta^n, \qquad Y_n = (y_{n,ij}),$$

$$Y'(\zeta) = \sum_{n=0}^{\infty} Y'_n \zeta^n, \qquad Y'_n = (y'_{n,ij}). \tag{6.4.14}$$

We assume that the multiplier matrix $X_-(\zeta)$ takes the form

$$X_-(\zeta) = 1 + \zeta^{-1} R. \tag{6.4.15}$$

Substututing (6.4.12), (6.4.14) and (6.4.15) into (6.4.13), we have

$$\begin{aligned} Y'(\zeta) = {} & \zeta^{-2} R Y_0 E_2 + \zeta^{-1}(RY_1E_2 + Y_0E_2) + Y_1E_2 \\ & + R(Y_0E_1 + Y_2E_2) + \sum_{n=1}^{\infty}\{Y_{n-1}E_1 + Y_{n+1}E_2 \\ & + R(Y_nE_1 + Y_{n+2}E_2)\}\zeta^n. \end{aligned}$$

The condition that the coefficients of ζ^{-2} and ζ^{-1} in the above expansion should vanish determines the 2×2 matrix R. Then the elements of $Y'(\zeta)$ is given by $Y(\zeta)$ directly. Summarizing these discussions, we prove

Proposition 6.4.2. *Let us set*

$$\tau = y_{0,12}y_{1,22} - y_{1,12}y_{0,22}. \tag{6.4.16}$$

Then the matrix R *of* (6.4.15) *is given by*

$$R = \tau^{-1}\begin{pmatrix} y_{0,12}y_{0,22} & -(y_{0,12})^2 \\ (y_{0,22})^2 & -y_{0,12}y_{0,22} \end{pmatrix}. \tag{6.4.17}$$

The elements $y'_{n,ij}$ *of* Y'_n *are expressed as follows:*

$$\begin{aligned}
y'_{0,11} &= \tau^{-1}y_{0,12}, \\
y'_{n,11} &= y_{n-1,11} + \tau^{-1}y_{0,12}(y_{0,22}y_{n,11} \\
&\qquad - y_{0,12}y_{n,21}), \\
y'_{0,21} &= \tau^{-1}y_{0,22}, \\
y'_{n,21} &= y_{n-1,21} + \tau^{-1}y_{0,22}(y_{0,22}y_{n,11} \\
&\qquad - y_{0,12}y_{n,21}), \\
y'_{n-1,12} &= y_{n,12} + \tau^{-1}y_{0,12}(y_{0,22}y_{n+1,12} \\
&\qquad - y_{0,12}y_{n+1,22}), \\
y'_{n-1,22} &= y_{n,22} + \tau^{-1}y_{0,22}(y_{0,22}y_{n+1,12} \\
&\qquad - y_{0,12}y_{n+1,22}), \quad n \geq 1.
\end{aligned} \tag{6.4.18}$$

The proof is given by direct calculaticn. The Riemann-Hilbert transformation: $Y \to Y'$, induced by (6.4.12), is referred to as the Bäcklund transformation.

Let $Y^{(1)}(\zeta)$ be a fundamental solution matrix corresponding to the ansatz $A_1^{(0)}$. Following [6], we obtain immediately the solution $Y^{(1)}(\zeta)$.

Lemma 6.4.3. *Let* $Y_n^{(1)} = (y_{n,ij}^{(1)})$ *be the coefficients of the Taylor expansion of* $Y^{(1)}(\zeta)$ *such that*

$$Y^{(1)}(\zeta) = \sum_{n=0}^{\infty} Y_n^{(1)} \zeta^n$$

Then $Y_n^{(1)}$ *can be expressed as*

$$y_{0,11}^{(1)} = -\Delta_{-1}/\Delta_0, \quad y_{1,11}^{(1)} = 1, \quad y_{n+2,11}^{(1)} = 0,$$

$$y_{0,21}^{(1)} = 1/\Delta_0, \quad y_{n+1,21}^{(1)} = 0,$$

$$y_{n,12}^{(1)} = \begin{vmatrix} \Delta_{-1} & \Delta_n \\ \Delta_0 & \Delta_{n+1} \end{vmatrix} / \Delta_0,$$

$$y_{n,22}^{(1)} = -\Delta_{n+1}/\Delta_0, \quad n \geq 0. \tag{6.4.19}$$

The proof is given by setting

$$\begin{pmatrix} \zeta & \psi(\zeta) \\ 0 & \zeta^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\alpha\beta - \gamma\delta = ad - bc = 1$ and $\alpha, \beta, \gamma, \delta$ are analytic except at $\zeta = \infty$, while a, b, c, d are analytic except at

$\zeta = 0$. The expansion (6.4.7) determines these elements under suitable gauge fixing. Then we derive form (6.4.5)

$$Y^{(1)}(\zeta) = \begin{pmatrix} d(\infty) & -b(\infty) \\ -c(\infty) & a(\infty) \end{pmatrix} \begin{pmatrix} \alpha(\zeta) & \beta(\zeta) \\ \gamma(\zeta) & \delta(\zeta) \end{pmatrix}.$$

This gives (6.5.19) in effect.

From Lemma 6.4.3, we have for $\zeta = 0$ the self-dual solution, after a change of the form,

$$P_1 = \Delta_0^{-1} \begin{pmatrix} 1 & -\Delta_1 \\ \Delta_{-1} & \Delta_0{}^2 - \Delta_{-1}\Delta_1 \end{pmatrix},$$

which is gauge equivalent to $P_1 = Y_0^{(1)}$. The factorization (6.1.18) gives

$$e = \Delta_{-1}, \quad f = \Delta_0, \quad g = -\Delta_1.$$

Then the condition (6.5.9) implies

$$f_y = g_{\bar{z}}, \quad f_z = -g_{\bar{y}}, \quad f_{\bar{y}} = -e_z, \quad f_{\bar{z}} = e_y.$$

From this we conclude that the ansatz $A_1^{(0)}$ is equivalent to the 't Hooft ansatz (6.3.11). In the next stage, we apply the Riemann-Hilbert transformation induced by (6.4.12) to the solution $Y^{(1)}(\zeta)$ corresponding to the ansatz $A_1^{(0)}$.

Let us define a sequence of $Y^{(N)}$ through the successive Riemann-Hilbert transformations

$$Y^{(N+1)}(\zeta) = u(\zeta)^{\circ}\, Y^{(N)}(\zeta), \quad N \geq 1. \qquad (6.4.20)$$

Here $u(\zeta)^{\circ}$ denote the Bäcklund transformation induced by (6.4.12). From the assumption, $Y^{(N)}$ are analytic near $\zeta = 0$. In general, the coefficients of the Taylor expansion of $Y^{(N)}$ can be expressed as the ratio of determinants with entries composed of $\{\Delta_n\}$. We have indeed the main theorem in this section.

Theorem 6.4.4. *Let* $Y_n^{(N)} = (y_{n,ij}^{(N)})$ *be the coefficients of the Taylor expansion of* $Y^{(N)}(\zeta)$*:*

$$Y^{(N)}(\zeta) = \sum_{n=0}^{\infty} Y_n^{(N)} \zeta^n. \qquad (6.4.21)$$

Then $Y_n^{(N)}$ *take the form*

$$y_{n,11}^{(N)} = (-1)^{N+n} \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{N-n-2} & \hat{\Delta}_{N-n-1} & \Delta_{N-n} & \cdots & \Delta_{N-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{N-2} & \cdots & \Delta_{2N-n-3} & \hat{\Delta}_{2N-n-2} & \Delta_{2N-n-1} & \cdots & \Delta_{2N-2} \end{vmatrix}$$

$$/D_{N-1}^{(N-1)}, \quad 0 \leq n \leq N, \quad y_{n,11}^{(N)} = 0, \quad N+1 \leq n, \qquad (6.4.22)$$

$$y_{n,21}^{(N)} = (-1)^{N+n+1} \begin{vmatrix} \Delta_{1} & \cdots & \Delta_{N-n-1} & \hat{\Delta}_{N-n} & \Delta_{N-n+1} & \cdots & \Delta_{N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{N-1} & \cdots & \Delta_{2N-n-3} & \hat{\Delta}_{2N-n-2} & \Delta_{2N-n-1} & \cdots & \Delta_{2N-2} \end{vmatrix}$$

$$/D_{N-1}^{(N-1)}, \quad 0 \leq n \leq N-1, \quad y_{n,21}^{(N)} = 0, \quad N \leq n, \qquad (6.4.23)$$

$$y_{n,12}^{(N)} = (-1)^{N+1} \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{N-2} & \Delta_{N+n-1} \\ \vdots & & \vdots & \vdots \\ \Delta_{N-1} & \cdots & \Delta_{2N-2} & \Delta_{2N+n-1} \end{vmatrix} / D_{N-1}^{(N-1)}, \qquad (6.4.24)$$

$$y_{n,22}^{(N)} = (-1)^{N} \begin{vmatrix} \Delta_{1} & \cdots & \Delta_{N-1} & \Delta_{N+n} \\ \vdots & & \vdots & \vdots \\ \Delta_{N} & \cdots & \Delta_{2N-2} & \Delta_{2N+N-1} \end{vmatrix} / D_{N-1}^{(N-1)}, \qquad (6.4.25)$$

where $D_n^{(m)}$ *is defined by* (6.4.10), *and the symbol* $\hat{}$ *denotes that the designated column is got rid of.*

Before proceeding the proof, we prepare the following lemmas in linear algebra.

Lemma 6.4.5. ([7]) *Let* M *be an* $n\times n$ *matrix and* $D\binom{i_1 \cdots i_r}{k_1 \cdots k_r}$ *a minor determinant defined by striking out the* i_1*-th, ..., the* i_r*-th rows and the* k_1*-th, ..., the* k_r*-th columns of* M. *Then*

$$|M|\cdot D\binom{ij}{kl} = D\binom{i}{j}D\binom{j}{l} - D\binom{i}{l}D\binom{j}{k} \qquad (6.4.26)$$

holds for $i<j$, $k<l$.

Lemma 6.4.6. *The following identity holds for* $1 \le n \le N+1$:

$$\begin{vmatrix} a_{11} \cdots a_{1N} \\ \vdots \quad\quad \vdots \\ a_{N1} \cdots a_{NN} \end{vmatrix} \begin{vmatrix} a_{22} \cdots a_{2N-n} & \hat{a}_{2N-n+1} & a_{2N-n+2} \cdots a_{2N+1} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ a_{N2} \cdots a_{NN-n} & \hat{a}_{NN-n+1} & a_{NN-n+2} \cdots a_{NN+1} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{12} \cdots a_{1N+1} \\ \vdots \quad\quad \vdots \\ a_{N2} \cdots a_{NN+1} \end{vmatrix} \begin{vmatrix} a_{21} \cdots a_{2N-n} & \hat{a}_{2N-n+1} & a_{2N-n+2} \cdots a_{2N} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ a_{N1} \cdots a_{NN-n} & \hat{a}_{NN-n+1} & a_{NN-n+2} \cdots a_{NN} \end{vmatrix}$$

$$= \begin{vmatrix} a_{21} \cdots a_{2N} \\ \vdots \quad\quad \vdots \\ a_{N1} \cdots a_{NN} \end{vmatrix} \begin{vmatrix} a_{11} \cdots a_{1N-n} & \hat{a}_{1N-n+1} & a_{1N-n+2} \cdots a_{1N+1} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ a_{N1} \cdots a_{NN-n} & \hat{a}_{NN-n+1} & a_{NN-n+2} \cdots a_{NN+1} \end{vmatrix}$$

Lemma 6.4.6 follows from Lemma 6.4.5. Now we give

Proof of Theorem 6.4.4. We prove only (6.4.24), because the remaining equations (6.4. 22, 23, 25) can be proved in the same way. The proof is given by induction with respect to N. Suppose the l-th induction step where $N = l$ has been accomplished. Let us introduce τ_N by

$$\tau_N = y^{(N)}_{0,12} y^{(N)}_{1,22} - y^{(N)}_{1,12} y^{(N)}_{0,22}. \qquad (6.4.28)$$

Substituting (6.4.24) and (6.4.25) with $N = l$ into (6.4.28), we have

$$\tau_l = \left(\begin{vmatrix} \Delta_{-1} \cdots & \Delta_{l-2} & \Delta_l \\ \vdots & \vdots & \vdots \\ \Delta_{l-1} \cdots & \Delta_{2l-2} & \Delta_{2l} \end{vmatrix} \cdot D^{(l)}_{l-1} \right.$$

$$- D_l^{(l-1)} \cdot \begin{vmatrix} \Delta_1 \cdots \Delta_{l-1} & \Delta_{l+1} \\ \vdots \quad\quad \vdots & \vdots \\ \Delta_l \cdots \Delta_{2l-2} & \Delta_{2l} \end{vmatrix} \Bigg) / (D_{l-1}^{(l-1)})^2 .$$

From Lemma 6.4.6, we obtain

$$\tau_l = D_l^{(l)} / D_{l-1}^{(l-1)} . \tag{6.4.29}$$

It should be noted that the following general formula can be given:

$$\begin{aligned} & y_{0,12}^{(l)} y_{n,22}^{(l)} - y_{n,12}^{(l)} y_{o,22}^{(l)} \\ & \quad = \begin{vmatrix} \Delta_0 \cdots \Delta_{l-1} & \Delta_{l+n-1} \\ \vdots \quad\quad \vdots & \vdots \\ \Delta_l \cdots \Delta_{2l-1} & \Delta_{2l+n-1} \end{vmatrix} / D_{l-1}^{(l-1)} . \end{aligned} \tag{6.4.30}$$

We proceed to the proof of (6.4.24) with $N = l+1$. Using Proposition 6.4.2, $y_{n,12}^{(l+1)}$ can be expressed as follows:

$$\begin{aligned} y_{n,12}^{(l+1)} &= y_{n+1,12}^{(l)} - \tau_l^{-1} y_{0,12}^{(l)} \\ & \quad \times (y_{0,12}^{(l)} y_{n+2,22}^{(l)} - y_{n+2,12}^{(l)} y_{0,22}^{(l)}) . \end{aligned} \tag{6.4.31}$$

Inserting (6.4.24) with $N = l$, and (6.4.30) with $n \to n+2$ into the right-hand side of (6.4.31), we derive

$$\tau_l y_{n,12}^{(l+1)} = (-1)^{l+1}\left(D_l^{(l)} \cdot \begin{vmatrix} \Delta_{-1} & \cdots \Delta_{l-2} & \Delta_{l+n} \\ \vdots & & \vdots \\ \Delta_{l-1} & \cdots \Delta_{2l-2} & \Delta_{2l+n} \end{vmatrix} \right.$$

$$\left. - D_l^{(l-1)} \cdot \begin{vmatrix} \Delta_0 \cdots & \Delta_{l-1} & \Delta_{l+n+1} \\ \vdots & \vdots & \vdots \\ \Delta_l \cdots & \Delta_{2l-1} & \Delta_{2l+n+1} \end{vmatrix} \right) / (D_{l-1}^{(l-1)})^2 .$$

From Lemma 6.4.5, we have (6.4.24) with $N = l+1$. This completes the proof. □

The fundamental solution matrix $Y^{(N)}(\zeta)$ given by Theorem 6.4.4 yields a solution of the $SL(2, \mathbb{C})$ self-dual equations (6.1.14) as $P^{(N)} = Y_0^{(N)} = (y_{0,ij}^{(N)})$, where each element is written as

$$y_{0,11}^{(N)} = (-1)^N D_{N-1}^{(N-2)} / D_{N-1}^{(N-1)} ,$$

$$y_{0,21}^{(N)} = (-1)^{N+1} D_{N-2}^{(N-1)} / D_{N-1}^{(N-1)} ,$$

$$y_{0,12}^{(N)} = (-1)^{N+1} D_N^{(N-1)} / D_{N-1}^{(N-1)} ,$$

$$y_{0,22}^{(N)} = (-1)^N D_{N-1}^{(N)} / D_{N-1}^{(N-1)} . \qquad (6.4.32)$$

Remark 6.4.7. Comparing (6.4.32) with $P_N^{(m)}$ given by (6.4.12), we see that $P^{(N)} = Y_0^{(N)}$ is equivalent to $P_N^{(N-1)}$.

The solutions $P^{(1)} = P_1^{(0)}$ and $P^{(N)} = P_N^{(N-1)}$ are corresponding to the ansatze $A_1^{(0)}$ and $A_N^{(N-1)}$, respectively. Under the successive Riemann-Hilbert transformations (6.4.20) from $N = 1$ to $N = l$ we have thus constructed the ansatz $A_{l+1}^{(l)}$ from $A_1^{(0)}$.

6.5. Reality conditions

As we have mentioned in Section 6.1, if a solution P of the self-dual equations (6.1.14) is a positive definite $SL(n, \mathbb{C})$ hermitian matrix for real x^μ, then the corresponding gauge potentials are real. Therefore, on physical grounds it is important to establish the transformation theory which keeps these reality conditions.

In this section, we discuss a degenerate (i.e., $u(\zeta) = 1$) Riemann-Hilbert transformation preserves the hermiticity of $GL(n, \mathbb{C})$ self-dual solutions. Belavin and Zahkarov [3] studies such a degenerate transformation to construct 't Hooft's instanton solutions. However, they treated $SL(2, \mathbb{C})$ case only. We also consider the positivity of our resulting solutions.

We work on the real Euclidean space determined by

$y = \bar{y}^*$ and $z = \bar{z}^*$. Let $Y(\zeta)$ be a fundamental solution matrix of (6.2.6) and $P = Y(0)$ be hermitian on the real Euclidean space. The degenerate Riemann-Hilbert transformation which we discuss takes the form

$$Y'(\zeta) = G(\zeta)Y(\zeta), \tag{6.5.1}$$

where $G(\zeta) = G(y, y^*, z, z^*; \zeta)$ is rational in ζ and is required to satisfy a normalization condition

$$G(\infty) = 1. \tag{6.5.2}$$

We refer to the Bäcklund transformation (2.2.6) for the sine-Gordon equation. This type of transformation is based upon the idea proposed by Zakharov and Mikhailov [14] and Belinski and Zakharov [4] in the study of two-dimensional field equations. First we look for a condition under which the solution given by

$$P' = Y'(0) \tag{6.5.3}$$

continues to be hermitian. The hermiticity of P' imposed a restriction on the analytical property of $G(\zeta)$ in the complex ζ-plane. We show the following proposition.

Proposition 6.5.1. *Let* $P = Y(0)$ *be hermitian. If* $G(\zeta)PG(-\zeta^{*-1})^\dagger$ *is independent of* ζ, *then one obtains for*

$P' = Y'(0)$

$$P' = G(\zeta)\, P\, G(-\zeta^{*-1})^{\dagger}, \qquad P' = P'^{\dagger}. \qquad (6.5.4)$$

Proof. From the assumption of the proposition, we have for $\zeta = 0$

$$G(\zeta)\, P\, G(-\zeta^{*-1})^{\dagger} = G(0)\, P\, G(\infty)^{\dagger}.$$

Since $G(\infty) = 1$ and $P' = G(0)P$, we prove the first assertion of (6.5.4). Similarly we obtain for $\zeta = \infty$

$$G(\zeta)\, P\, G(-\zeta^{*-1})^{\dagger} = G(\infty)\, P\, G(0)^{\dagger}.$$

From $G(\infty) = 1$, $P = G(0)P$ and $P = P^{\dagger}$, we prove $P' = P'^{\dagger}$. This completes the proof. $\square$

Let us consider the degenerate Riemann-Hilbert transformation associated with

$$G(\zeta) = 1 + \sum_{j=1}^{N_1} \frac{K_j}{\zeta - \zeta_j} + \sum_{j=1}^{N_2} \frac{L_j}{\zeta z^* - y - \zeta z_j^* + y_j} + \sum_{j=1}^{N_3} \frac{M_j}{\zeta y^* + z - \zeta y_{j+N_2}^* - z_{j+N_2}},$$

$$G(\zeta)^{-1} = 1 + \sum_{j=1}^{N_1} \frac{K_j'}{\zeta + \zeta_j^{*-1}} + \sum_{j=1}^{N_2} \frac{L_j'}{\zeta y^* + z - \zeta y_j^* - z_j}$$

$$+\sum_{j=1}^{N_3} \frac{M_j'}{\zeta z^* - y - \zeta z_{j+N_2}^* + y_{j+N_2}}, \tag{6.5.6}$$

where ζ_j, y_j and z_j are constants, where matrices K_j, L_j and M_j are to be determined. Let us set

$$\alpha_j = \frac{y - y_j}{z^* - z_j^*}, \quad \beta_j = -\frac{z - z_{j+N_2}}{y^* - y_{j+N_2}^*}. \tag{6.5.7}$$

We assume that $\zeta_j, -\zeta_j^{*-1}, \alpha_j, -\alpha_j^{*-1}, \beta_j$ and $-\beta_j^{*-1}$ are mutually distinct. From the proof of Theorem 2.2.2 in Chapter II, we have

Proposition 6.5.2. *We substutute* $Y'(\zeta)$ *given by* (6.5.1) *with* (6.5.6) *into the linear equations to derive*

$$D_k Y'(\zeta) \cdot Y'(\zeta)^{-1} = D_k G(\zeta) \cdot G(\zeta)^{-1} + \zeta^{-1} G(\zeta) A_k G(\zeta)^{-1}$$

If $D_k Y'(\zeta) \cdot Y'(\zeta)^{-1}$, $k = 1, 2$, *are analytic at the poles of* G *and* G^{-1}, $\zeta_j, -\zeta_j^{*-1}, \alpha_j, -\alpha_j^{*-1}, \beta_j$ *and* $-\beta_j^{*-1}$, *then the transformation* (6.5.1) *actually gives a solution* $P' = Y'(0)$ *of the self-dual equations* (6.1.14).

Since the residues

$$K_j G(\zeta_j)^{-1}, \quad L_j G(\alpha_j)^{-1}, \quad M_j G(\beta_j)^{-1}$$

of $G(\zeta)G(\zeta)^{-1}$ at $\zeta=\zeta_j$, α_j and β_j must vanish, K_j, L_j and M_j are degenerate matrices. We assume that these matrices are of rank 1 and take the form

$$K_j = T_j \cdot {}^tQ_j, \quad L_j = U_j \cdot {}^tR_j, \quad M_j = V_j \cdot {}^tS_j, \tag{6.5.8}$$

where Q_j, R_j, S_j, T_j, U_j and V_j are n-dimensional column vectors and ${}^tQ_j$ denotes the transpose of Q_j. We note that

$${}^tQ_jG(\zeta_j)^{-1} = {}^tR_jG(\alpha_j) = {}^tS_jG(\beta_j) = 0. \tag{6.5.9}$$

Using the condition given in Propositions 6.5.1 and 6.5.2, we have the following main theorem.

Theorem 6.5.3. *Suppose we have* $G(\zeta)$ *given by* (6.5.6) *with* (6.5.8). *Let us define* Q_j, R_j *and* S_j *by*

$${}^tQ_j = {}^tQ_jY(\zeta_j)^{-1}, \quad {}^tR_j = {}^tr_jY(\alpha_j)^{-1},$$

$${}^tS_j = {}^ts_jY(\beta_j)^{-1}, \tag{6.5.10}$$

where q_j, r_j *and* s_j *are column vectors such that*

$$D_{k(\zeta_j)}q_j = D_{k(\alpha_j)}r_j = D_{k(\beta_j)}\ j = 0. \tag{6.5.11}$$

Here the differential operators are introduced by

$$D_{1(\zeta_j)} = \zeta_j^{-1}\partial_{\bar{z}} + \partial_y, \quad D_{2(\zeta_j)} = \zeta_j^{-1}\partial_{\bar{y}} + \partial_z,$$

and so on. Set Q_j^* *as the complex conjugate vector of* Q_j. *And we define* T_j, U_j *and* V_j *by*

$$(T_1, \ldots, T_N, U_1, \ldots, U_N, \ldots, V_1, \ldots, V_N)Z$$
$$= (PQ_1^*, \ldots, PQ_N^*, PR_1^*, \ldots$$
$$\ldots, PR_N^*, PS_1^*, \ldots, PS_N^*) \qquad (6.5.12)$$

Here $Z = (Z^{(ij)})$, $1 \le i,j \le 3$, *is the* $3(N_1+N_2+N_3)\times 3(N_1+N_2+N_3)$ *matrix whose* (i,j)*-block* $Z^{(ij)} = (Z_{kl}^{(ij)})$, $1 \le k \le N_i$, $1 \le l \le N_j$, *is given by*

$$Z_{kl}^{(11)} = {}^tQ_k PQ_l^*/(\zeta_l^{*-1} + \zeta_k),$$
$$Z_{kl}^{(12)} = {}^tQ_k PR_l^*/(\alpha_l^{*-1} + \zeta_k),$$
$$Z_{kl}^{(13)} = {}^tQ_k PS_l^*/(\beta_l^{*-1} + \zeta_k),$$

$$Z_{kl}^{(21)} = {}^tR_k PQ_l^*/\{(z - z_k^*)(\zeta_l^{*-1} + \alpha_k)\},$$
$$Z_{kl}^{(22)} = {}^tR_k PR_l^*/\{(z - z_k^*)(\alpha_l^{*-1} + \alpha_k)\},$$
$$Z_{kl}^{(23)} = {}^tR_k PS_l^*/\{(z - z_k^*)(\beta_l^{*-1} + \alpha_k)\},$$

$$Z_{kl}^{(31)} = {}^tS_k PQ_l^*/\{(y - y_{N_2+k}^*)(\zeta_l^{*-1} + \beta_k)\},$$
$$Z_{kl}^{(32)} = {}^tS_k PR_l^*/\{(y - y_{N_2+k}^*)(\alpha_l^{*-1} + \beta_k)\},$$
$$Z_{kl}^{(33)} = {}^tS_k PS_l^*/\{(y - y_{N_2+k}^*)(\beta_l^{*-1} + \beta_k)\}. \qquad (6.5.13)$$

Then $P' = G(0)P$ *is a hermitian solution of the* $GL(n, \mathbb{C})$ *self-dual equations.*

Proof. To begin with, let us consider the condition of Proposition 6.5.2 that

$$D_k G(\zeta)\cdot G(\zeta)^{-1} + \zeta^{-1} G(\zeta) A_k G(\zeta)^{-1},$$

$k = 1, 2$, are analytic at $\zeta = \zeta_j$. This requirement is equivalent to

$$D_{k(\zeta_j)} K_j \cdot G(\zeta_j)^{-1} + \zeta_j^{-1} K_j A_k G(\zeta_j)^{-1} = 0. \qquad (6.5.13)$$

From (6.5.9) we note that

$$D_{k(\zeta_j)} K_j \cdot G(\zeta_j)^{-1} = T_j \cdot D_{k(\zeta_j)} {}^tQ_j \; G(\zeta_j)^{-1}.$$

Then we see a sufficient condition for (6.5.13) is given by

$$D_{k(\zeta_j)} {}^tQ_j + \zeta_j^{-1} \; {}^tQ_j A_k = 0. \qquad (6.5.14)$$

On the other hand, from (6.2.6),

$$D_{k(\zeta_j)} Y(\zeta_j)^{-1} + \zeta_j^{-1} Y(\zeta_j)^{-1} A_k = 0. \qquad (6.5.15)$$

Comparing (6.5.14) with (6.5.15), we see that the Q_j defined by the first equation of (6.5.10) meets the sufficient condition (6.5.14). The R_j and S_j defined by the second and third equations of (6.5.10), respectively, meet also (6.5.14). This is seen from the regularity at $\zeta = \alpha_j,\ \beta_j$ of $D_k G \cdot G^{-1} + \zeta^{-1} G A_k G^{-1}$.

We turn to (6.5.12). The condition of Proposition 6.5.1 is satisfied if $G(\zeta)PG(-\zeta^{*-1})^{\dagger}$ is analytic at $\zeta = -\zeta_j^{*-1},\ -\alpha_j^{*-1}$ and $-\beta_j^{*-1}$. To see this we set

$$\operatorname*{Res}_{\zeta=-\zeta_j^{*-1}} G(\zeta)PG(-\zeta^{*-1})^{\dagger} = G(-\zeta_j^{*-1})PK_j^{\dagger} = 0,$$

$$G(-\alpha_j^{*-1})PL_j^{\dagger} = G(-\beta_j^{*-1})PM_j^{\dagger} = 0, \tag{6.5.16}$$

respectively. If Equations (6.5.16) hold, $G(\zeta)PG(-\zeta^{*-1})^{\dagger}$ is consequently analytic at $\zeta = \zeta_j,\ \alpha_j,\ \beta_j$ because P is hermitian. From Liouville's theorem, it follows that $G(\zeta)PG(-\zeta^{*-1})^{\dagger}$ is independent of ζ. The first equation of (6.5.16) yields

$$\sum_{k=1}^{N} \frac{{}^tQ_k PQ_j^*}{\zeta_j^{*-1} + \zeta_k} T_k + \sum_{k=1}^{N_2} \frac{{}^tR_k PQ_j^*}{(z^* - z_k^*)(\zeta_j^{*-1} - \alpha_k)} U_k$$

$$+ \sum_{k=1}^{N_3} \frac{{}^tS_k PQ_j^*}{(y^* + y_{N_2+k}^*)(\zeta_j^{*-1} + \beta_k)} V_k = PQ_j^*.$$

The second and the third equations of (6.5.16) also give rise to similar equations as the above. Then we obtain (6.5.12) with (6.5.13). Thus Proposition 6.5.1 gurantees that P' given by (6.5.4) is hermitian.

To accomplish the proof, we have to show that $D_kG\cdot G^{-1}+\zeta^{-1}GA_kG^{-1}$ are analytic at $\zeta = -\zeta_j^{*-1}, -\alpha_j^{*-1}, -\beta_j^{*-1}$ (see Proposition 6.5.2). For this purpose, we prepare the following lemma.

Lemma 6.5.4. *For simplicity we set*

$$E_k(\zeta) = D_kG(\zeta)\cdot G(\zeta)^{-1} + \zeta^{-1}G(\zeta)A_kG(\zeta)^{-1},$$

$k = 1, 2$. *If* $E_k(\zeta)$ *are analytic at* $\zeta = \zeta_j, \alpha_j, \beta_j$ *and* $G(\zeta)PG(-\zeta^{*-1})^\dagger$ *is analytic at* $\zeta = \zeta_j, -\zeta_j^{*-1}, \alpha_j, -\alpha_j^{*-1}, \beta_j\ -\beta_j^{*-1}$, *then* $E_k(\zeta)$ *are analytic at* $\zeta = -\zeta_j^{*-1}, -\alpha_j^{*-1}$, *and* $-\beta_j^{*-1}$.

Proof of Lemma 6.5.4. Multiplying $E_k(\zeta)$ by $P' = G(\zeta)PG(-\zeta^{*-1})^\dagger$, we obtain

$$\begin{aligned} E_k(\zeta)P' = {} & D_kG(\zeta)\cdot PG(-\zeta^{*-1})^\dagger \\ & + \zeta^{-1}G(\zeta)\partial_{\bar{k}}P\, G(-\zeta^{*-1})^\dagger, \end{aligned} \tag{6.5.17}$$

$k = 1, 2$, where $\partial_{\bar{1}}P = \partial_{\bar{z}}P$ and $\partial_{\bar{2}}P = \partial_{\bar{y}}P$. On the other hand, operating on P' with D_k, we have

$$\begin{aligned} D_kP' = {} & D_kG(\zeta)\cdot PG(-\zeta^{*-1})^\dagger + G(\zeta)D_kP\cdot G(-\zeta^{*-1})^\dagger \\ & + G(\zeta)PD_kG(-\zeta^{*-1})^\dagger. \end{aligned} \tag{6.5.18}$$

Equations (6.5.17) and (6.5.18) give

$$E_k(\zeta)P' = D_kP' - G(\zeta)\partial_k P\, G(-\zeta^{*-1})^\dagger$$
$$- G(\zeta)PD_kG(-\zeta^{*-1})^\dagger, \qquad (6.5.19)$$

where $\partial_1 P = \partial_y P$ and $\partial_2 P = -\partial_z P$. From the assumption, $E_k(\zeta)P'$ is analytic at $\zeta = \zeta_j$, α_j and β_j. Let us take the hermitian conjugate of (6.5.19) with $k=1$ and replace ζ by $-\zeta^{*-1}$. From $P = P^\dagger$, we see that

$$D^*_{1(-\zeta^{*-1})}G(\zeta)\cdot PG(-\zeta^{*-1})^\dagger + G(\zeta)\partial_{\bar{y}} P\, G(-\zeta^{*-1})^\dagger$$

is analytic at $\zeta = -\zeta_j^{*-1}$, $-\alpha_j^{*-1}$ and $-\beta_j^{*-1}$, where

$$D^*_{1(-\zeta^{*-1})} = (\zeta^{*-1}\partial_z + \partial_{\bar{y}})\Big|_{\zeta \to -\zeta^{*-1}} = \zeta D_2.$$

Then $E_2(\zeta)P'$ of (6.5.17) with $k=2$ is analytic at $\zeta = -\zeta_j^{*-1}$, $-\alpha_j^{*-1}$ and $-\beta_j^{*-1}$. We can prove also the regularity of $E_1(\zeta)P'$ of (6.5.17) with $k=1$ by the same manner. This proves the lemma. □

Making use of results given in Lemma 6.5.4, we conclude that each $E_k(\zeta)$ has a simple pole at $\zeta = 0$. Hence we can take $E_k(\zeta)$ to be $\zeta^{-1}A'_k$. Now we have

$$D_kY'(\zeta) = \zeta^{-1}A'_k\, Y'(\zeta),$$

and therefore from the compatibility condition A'_k can be put in the form

$$A'_1 = \partial_{\bar{z}} P' \cdot P'^{-1}, \qquad A'_2 = \partial_{\bar{y}} P' \cdot P'^{-1},$$

where P' is given by $P' = G(0)P$. Thus P' becomes a solution of the $GL(n, \mathbb{C})$ self-dual equations. This completes the proof of Theorem 6.5.3. □

Finally we discuss the positivity of the resulting solution P'. We restrict ourselves to the case $n = 2$. Since the coefficient matrices K_j, L_j and M_j in $G(\zeta)$ are degenerate, the determinant of $G(\zeta)$ at $\zeta = 0$ becomes

$$(-1)^{N_1+N_2+N_3} \prod_{j=1}^{N_1} |\zeta_j|^{-2} \prod_{j=1}^{N_2} |\alpha_j|^{-2} \prod_{j=1}^{N_3} |\beta_j|^{-2}.$$

Due to this, $\det P$ and $\det P'$ are of the same sign when $N_1 + N_2 + N_3$ is even, but not so are they when $N_1 + N_2 + N_3$ is odd. Then if P is positive definite, P' is definite, positive or negative, when $N_1 + N_2 + N_3$ is even. On the contrary, the odd transformation (6.5.1), where $N_1 + N_2 + N_3$ is odd, violates the positivity of solutions.

CHAPTER VII

SYMMETRIES OF THE SELF-DUAL EQUATIONS

Over the past decate, considerable progress has been made in understanding the symmetry of nonlinear fields. Indeed, for the stationary axially symmetric gravitational fields, Geroch [5] showed that the Ehlers transformation can be employed to construct an infinite-dimensional transformation group called the Geroch group (see Section 5.1). Kinnersley and Chitre [7, 8] gave a useful expression of the infinitesimal transformations of the Geroch group. The existence of such transformation group is essentially due to the Ehlers transformation and the duality of the factrization (see (3.1.4) and (4.3.1)).

It is of great interest to generalize the Geroch, Kinnersley and Chitre formalism to other field equations, especially in four-dimensional space. Recently, Maison [9] has showed that stationary Jordan's equations in five-dimensional unified theory admit an Ehlers-type transformation. He also has derived Noether currents with respect to the transformation.

The first purpose of this chapter is to propose an Ehlers-type transformation for $SL(2, \mathbb{C})$ self-dual equations of $SU(2)$ Yang-Mills gauge fields in four-dimensional Euclidean space. Noether currents are given for the 't Hooft ansatz.

Using the Ehlers-type transformation and a factrization dual to Yang's R gauge [13], we derive three independent infinitesimal transformations for $SL(2, \mathbb{C})$ self-dual equations, which are analogous to the Kinnersley-Chitre transformations of the Geroch group. This is the second purpose of the chapter.

As was mentioned in Section 5.1, the infinitesimal transformations of the Geroch group form the infinite-dimensional Lie algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ which is called *the symmetry algebra* of the stationary axially symmetric Einstein equations. Recently, symmetry algebras acting on the other nonlinear field equations have been investigated from the viewpoints of variational calculus [2] and the Riemann-Hilbert problem [12]. The interrelation of these methods will be studied in Section 8.3. The third purpose is to present an infinite number of symmetries and a symmetry algebra of the $GL(n, \mathbb{C})$ self-dual equations.

In Section 7.1, an Ehlers-type transformation group H is introduced through the $SU(2)$ R gauge factrization. We propose a $SU(1,1)$ factrization dual to the R gauge. In Section 7.2, first the infinitesimal transformation generated by the group H is given so as to act on the factors in the $SU(1,1)$ factrization. Next we consider the second symmetry group G. Then the products of H and G yield other two independent infinitesimal transformations. In Section 7.3, an infinite number of nonlocal symmetries of the self-dual equations is presented by making use of the Zakharov-Shabat system (see (6.2.6)). Finally, in Section 7.4, we discuss the infinitesimal Riemann-Hilbert transformations.* The resulting infinitesimal transformation group is isomorphic to the Lie algebra $gl(n, \mathbb{C}) \otimes \mathbb{C}[w, w^{-1}, w_1, w_2]$.

7.1. Ehlers-type transformation

Ehlers [3] discovered a surprising transformation for the stationary vacuum gravitational fields. The transformation called *the Ehlers* $SL(2, \mathbb{R})$ *rotation* give rise to a one-parameter family of physically different solutions from any stationary solution. In this section, we present Ehlers-type transformation which may form a Geroch-like transformation group.

By analytic continuation of gauge potentials into the complex space $\mathbb{C}^4$, the $SU(2)$ self-dual Yang-Mills equations $*F_{\mu\nu} = F_{\mu\nu}$ are written as

---

* Main part of Section 7.4 is the collaboration with Lecturer Kimio UENO.

$$\partial_y(\partial_{\bar{y}} P \cdot P^{-1}) + \partial_z(\partial_{\bar{z}} P \cdot P^{-1}) = 0, \tag{7.1.1}$$

here $(y, \bar{y}, z, \bar{z}) \in \mathbb{C}^4$ and $P \in SL(2, \mathbb{C})$, (see Section 6.1). These equations are called the $SL(2, \mathbb{C})$ self-dual equations. If a solution P is positive definite and hermitian, $SU(2)$ real gauge fields are derived from P. We factrize P in the form

$$P = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2 + eg \end{pmatrix}. \tag{7.1.2}$$

We note here that the P admits another factrization:

$$P = \frac{1}{2f}\begin{pmatrix} f^2+(1+e)(1+g) & f^2-(1+e)(1-g) \\ f^2-(1-e)(1+g) & f^2+(1-e)(1-g) \end{pmatrix}. \tag{7.1.3}$$

The self-dual equation (7.1.1) are written out from (7.1.3) as well as (7.1.2) to be the Yang equations

$$f\Box f - f_y f_{\bar{y}} - f_z f_{\bar{z}} + e_y g_{\bar{y}} + e_z g_{\bar{z}} = 0, \tag{7.1.4}$$

$$f\Box e - 2e_y f_{\bar{y}} - 2e_z f_{\bar{z}} = 0,$$

$$f\Box g - 2g_{\bar{y}} f_y - 2g_{\bar{z}} f_z = 0, \tag{7.1.5}$$

where $\Box = \partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}}$. The factrization (7.1.2) is the R gauge (6.1.18), while (7.1.3) is a new factrization.

Let h be an $SL(2, \mathbb{C})$ constant matrix. The it is obvious that the self-dual equations (7.1.1) are invariant under the transformation

$$h:\ P \to hPh^{\dagger}. \tag{7.1.6}$$

We call this transformation group *group* H after the work of Kinnersley and Chitre [7, 8] on two-dimensional gravitational fields. Denote the generators of group H by

$$h_a(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad h_b(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad h_c(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

where λ is a complex parameter. It should be noted that the actions of group H take a simple form for the R gauge. We have

$$h_a(\lambda):\ \begin{pmatrix} e \\ f \\ g \end{pmatrix} \to \frac{1}{\Gamma}\begin{pmatrix} e + \lambda^*(f^2 + eg) \\ f \\ g + \lambda(f^2 + eg) \end{pmatrix}, \tag{7.1.7}$$

$$h_b(\lambda):\ \begin{pmatrix} e \\ f \\ g \end{pmatrix} \to \begin{pmatrix} e + \lambda \\ f \\ g + \lambda^* \end{pmatrix},$$

$$h_c(\lambda):\ \begin{pmatrix} e \\ f \\ g \end{pmatrix} \to \begin{pmatrix} \lambda^2 e \\ |\lambda|^2 f \\ \lambda^{*2} g \end{pmatrix}, \tag{7.1.8}$$

where $\Gamma = 1 + \lambda e + \lambda^* g + |\lambda|^2(f^2 + eg)$. The actions (7.1.8) are a gauge transformation and a scale transformation, respectively. The action (7.1.7) induced from $h_a(\lambda)$ is analogous to the Ehlers transformation (5.1.4) in the general relativity.

The simplest instanton solution of the $SU(2)$ self-dual equations is given by

$$e = f = -g = \frac{y\bar{y} + z\bar{z}}{1 + y\bar{y} + z\bar{z}}$$

(see (6.3.10)). The Ehlers-type transformation $h_a(\lambda)$ yields from the above the solution

$$e = f = -g = \frac{y\bar{y} + z\bar{z}}{1 + y\bar{y} + z\bar{z} + (\lambda - \lambda^*)(y\bar{y} + z\bar{z})},$$

which violates the reality of gauge potentials.

The Yang equations (7.1.4) and (7.1.5) are the Euler equations for the variational problem with the Lagrangian density

$$L = \frac{1}{f^2}(f_y f_{\bar{y}} + f_z f_{\bar{z}} + e_y g_{\bar{y}} + e_z g_{\bar{z}}).$$

The change in L under the infinitesimal Ehlers-type transformation is not equal to zero except for the case of $e = g$ and $\lambda = \lambda^*$. This corresponds to the well-known 't Hooft ansatz (see (6.3.11)). We set J_α as the variation of L:

$$J_\alpha \lambda = \frac{\delta L}{\delta f_\alpha} \delta_a f + \frac{\delta L}{\delta e_\alpha} \delta_a e, \qquad (7.1.9)$$

where $f_\alpha = \partial_\alpha f$, $e_\alpha = \partial_\alpha e$, $\alpha = y, \bar{y}, z, \bar{z}$ and $\delta_a f$ and $\delta_a e$ are small increments of f and e under $h_a(\lambda)$, respectively. The equations

$$\partial_y J_y + \partial_z J_z = 0, \quad \partial_{\bar{y}} J_{\bar{y}} + \partial_{\bar{z}} J_{\bar{z}} = 0 \tag{7.1.10}$$

can be verified with the help of the Yang equations. Thus the functions J_α play the role of Noether currents with respect to the Ehlers-type transformation.

Next, we introduce a factrization dual to the R gauge (7.1.2). It is not difficult to prove that the field equations (7.1.5) imply the existence of new dependent variables $\varepsilon = \varepsilon(y, \bar{y}, z, \bar{z})$ and $\kappa = \kappa(y, \bar{y}, z, \bar{z})$ satisfying

$$\varepsilon_y = f^{-2} g_{\bar{z}}, \quad \varepsilon_z = - f^{-2} g_{\bar{y}},$$

$$\kappa_{\bar{y}} = - f^{-2} e_z, \quad \kappa_{\bar{z}} = f^{-2} e_y. \tag{7.1.11}$$

Therefore it is shown that ε, f and κ satisfy the following system:

$$f \square f - f_y f_{\bar{y}} - f_z f_{\bar{z}} - f^4 \varepsilon_y \kappa_{\bar{y}} - f^4 \varepsilon_z \kappa_{\bar{z}} = 0, \tag{7.1.12}$$

$$f \square e + 2\varepsilon_y f_{\bar{y}} + 2\varepsilon_z f_{\bar{z}} = 0,$$

$$f \square g + 2\kappa_y f_{\bar{y}} + 2\kappa_z f_{\bar{z}} = 0. \tag{7.1.13}$$

Let $Q = Q(y, \bar{y}, z, \bar{z})$ be a unimodular matrix defined by

$$Q = \frac{1}{f}\begin{pmatrix} f^2 & f^2\kappa \\ f^2\varepsilon & 1 + f^2\varepsilon\kappa \end{pmatrix}. \tag{7.1.14}$$

Then we can prove that Equations (7.1.12) and (7.1.13) are defived from

$$\partial_y(\partial_{\bar{y}} Q\cdot Q^{-1}) + \partial_z(\partial_{\bar{z}} Q\cdot Q^{-1}) = 0. \tag{7.1.15}$$

These are also the $SL(2, \mathbb{C})$ self-dual equations. We have

Remark 7.1.1. There is a formal mapping

$$I: \begin{pmatrix} \varepsilon \\ f \\ \kappa \end{pmatrix} \to \begin{pmatrix} e \\ f^{-1} \\ g \end{pmatrix} \tag{7.1.16}$$

which transforms the system (7.1.12) and (7.1.13) to the Yang equations (7.1.4) and (7.1.5).

We call (7.1.14) *the dual factrization* of the R gauge. Since the reality condition $e = g^*$ for the R gauge turns into $\varepsilon = -\kappa^*$, Equations (7.1.15) describe $SU(1,1)$ gauge fields.

7.2. Infinitesimal transformations

First, we consider the Ehlers-type transformation for the dual factrization (7.1.14). Let us introduce the matrix functions Φ and Ψ determined by

$$\partial_y \Phi = -\partial_{\bar{z}} P \cdot P^{-1}\sigma , \quad \partial_z \Phi = \partial_{\bar{y}} P \cdot P^{-1}\sigma,$$

$$\partial_{\bar{y}} \Psi = -\sigma P^{-1}\partial_z P, \quad \partial_{\bar{z}} \Psi = \sigma P^{-1}\partial_y P, \qquad (7.2.1)$$

where $\Phi = (\phi_{ij})$, $\Psi = (\psi_{ij})$, $\sigma = (\varepsilon_{ij})$, $i, j = 1, 2$ and ε_{ij} are the skew-symmetric tensors with $\varepsilon_{12} = 1$. Using the action (7.1.7), we can prove that Φ and Ψ are subject to change under the transformation $h_a(\lambda)$ as

$$h_a(\lambda) : \Phi \to \Phi + \begin{pmatrix} \lambda^*\phi_{12} + \lambda\phi_{21} + |\lambda|^2\phi_{22} & \lambda\phi_{22} \\ \lambda^*\phi_{22} & 0 \end{pmatrix},$$

$$h_a(\lambda) : \Psi \to \Psi + \begin{pmatrix} \lambda^*\psi_{12} + \lambda\psi_{21} + |\lambda|^2\psi_{22} & \lambda\psi_{22} \\ \lambda^*\psi_{22} & 0 \end{pmatrix}, \qquad (7.2.2)$$

Here we have neglected the integration constants. Then we have

Lemma 7.2.1. *Under the transformation* $h_a(\lambda)$, *the functions* ε *and* κ *are transformed as*

$$h_a(\lambda) : \varepsilon \to \varepsilon + \lambda^*\phi_{12} + \lambda\phi_{21} + |\lambda|^2\phi_{22},$$

$$h_a(\lambda): \kappa \to \kappa + \lambda^*\psi_{12} + \lambda\psi_{21} + |\lambda|^2\psi_{22}. \qquad (7.2.3)$$

Proof. From (7.1.7) and (7.2.1), we obtain

$$\partial_y \phi_{11} = f^{-2} g_{\bar{z}}, \qquad \partial_z \phi_{11} = - f^{-2} g_{\bar{y}},$$

$$\partial_{\bar{y}} \psi_{11} = - f^{-2} e_z, \qquad \partial_{\bar{z}} \psi_{11} = f^{-2} e_y.$$

These equations with (7.1.11) imply $\phi_{11} = \varepsilon$ and $\psi_{11} = \kappa$. Hence the actions (7.2.2) give (7.2.3). This proves the lemma. □

Let Θ and Ω be 2×2 matrices determined by

$$\partial_y \Theta = - \partial_{\bar{z}} Q \cdot Q^{-1} \sigma, \qquad \partial_z \Theta = \partial_{\bar{y}} Q \cdot Q^{-1} \sigma,$$

$$\partial_{\bar{y}} \Omega = - \sigma Q^{-1} \partial_z Q, \qquad \partial_{\bar{z}} \Omega = \sigma Q^{-1} \partial_y Q, \qquad (7.2.4)$$

where $\Theta = (\theta_{ij})$ and $\Omega = (\omega_{ij})$. From these definitions, we see $\theta_{11} = e$ and $\omega_{11} = g$. It is not difficult to prove

Lemma 7.2.2. *The algebraic relations*

$$\phi_{12} = \phi_{21}, \quad \psi_{12} = \psi_{21}, \quad \theta_{12} = \theta_{21}, \quad \omega_{12} = \omega_{21},$$

$$\phi_{12} + \theta_{12} = e\varepsilon, \quad \psi_{12} + \omega_{12} = g\kappa \qquad (7.2.5)$$

hold.

We have not succeeded in expressing ϕ_{22} and ψ_{22} in terms of P, Q, Θ or Ω. This does not prevent us from considering the infinitesimal $h_a(\lambda)$ transformation for Q. We have

Proposition 7.2.3. *The Ehlers-type transformation* $h_a(\lambda)$ *makes an infinitesimal change in* Q *as follows:*

$$h_a(\lambda):\ Q \to Q - \delta_a Q,$$

$$\delta_a q_{11} = (\lambda\theta_{11} + \lambda^*\omega_{11})q_{11},$$

$$\delta_a q_{12} = (\lambda + \lambda^*)\omega_{12}q_{11} + \lambda(\theta_{11} - \omega_{11})q_{12},$$

$$\delta_a q_{21} = (\lambda + \lambda^*)\theta_{12}q_{11} - \lambda^*(\theta_{11} - \omega_{11})Q_{21},$$

$$\delta_a q_{22} = (\lambda + \lambda^*)(\theta_{12}q_{12} + \omega_{12}q_{21}) - (\lambda^*\theta_{11} + \lambda\omega_{11})q_{22} - (\lambda - \lambda^*)(\theta_{11} - \omega_{11})q_{11}^{-1}, \qquad (7.2.6)$$

where λ is considered as an infinitesimal parameter.

Proof. Using Lemmas 7.2.1 and 7.2.2, we derive for the infinitesimal parameter λ

$$h_a(\lambda):\varepsilon \to \varepsilon + (\lambda + \lambda^*)\phi_{12} = \varepsilon + (\lambda + \lambda^*)(\varepsilon\theta_{11} - \theta_{12}),$$

$$h_a(\lambda):f \to f - (\lambda e + \lambda^* g)f = f - (\lambda\theta_{11} + \lambda^*\omega_{11})f,$$

$$h_a(\lambda): \kappa \to \kappa + (\lambda + \lambda^*)\psi_{12} = \kappa + (\lambda + \lambda^*)(\kappa\omega_{11} - \omega_{12}).$$

These give the infinitesimal transformation (7.2.6). $\square$

In Section 7.1 we have considered an $SL(2, \mathbb{C})$ rotation group H for the self-dual equations (7.1.1). In similar manner, we can work with an $SL(2, \mathbb{C})$ rotation group for the self-dual equations (7.1.15). We call it *group* G, especially. The group G plays the same role as Kinnersley's G group [8] for the gravitational fields. His group describes $SL(2, \mathbb{R})$ covariance under the coordinate transformations, and corresponds to the external symmetry of the Einstein equations, compared with the internal or hidden symmetry (5.1.3). We write the action and generators of group G as

$$g: Q \to gQg^{\dagger}, \tag{7.2.7}$$

$$g_a(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad g_b(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad g_c(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

Next, we consider the combination of group G with group H in the form

$$G^{-1} \circ I^{-1} \circ H \circ I \circ G, \tag{7.2.8}$$

where I is the mapping $I: Q \to P$ defined by (7.1.16).

When $h_a(\lambda) \in H$ is assigned, the composition (7.2.8) is most interenting. If $g_1 = 1 \in G$ is taken, the infinitesimal transformation $g_1^{-1} \circ I^{-1} \circ h_a(\lambda) \circ I \circ g_1$ reduces to (7.2.6). Here and in the sequel, λ is the infinitesimal parameter. For simplicity, we set

$$K_1(\lambda_1) = g_1^{-1} \circ I^{-1} \circ h_a(\lambda_1) \circ I \circ g_1. \tag{7.2.9}$$

Other infinitesimal transformations independent of $K_1(\lambda)$ are obtained as follows:

$$K_2(\lambda_2) = g_2^{-1} \circ I^{-1} \circ h_a(\lambda_2) \circ I \circ g_2, \tag{7.2.10}$$

$$\begin{aligned} K_3(\lambda_3) = {} & K_2(-\lambda_3) \circ K_1(-\lambda_3) \circ g_3^{-1} \circ I^{-1} \\ & \circ h_a(\lambda_3) \circ I \circ g_3, \end{aligned} \tag{7.2.11}$$

$$\text{with} \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We restrict ourselves to the case of $\lambda_k = \lambda_k^*$, $k = 1, 2, 3$. Making use of calculation performed in Proposition 7.2.3, we have

Proposition 7.2.4. *Let* $\delta_2 Q$ *and* $\delta_3 Q$ *be the infinitesimal changes in* Q *under the transformations* $K_2(\lambda_2)$ *and* $K_3(\lambda_3)$, *respectively. Then* $\delta_2 Q$ *and* $\delta_3 Q$ *take the form*

$$\delta_2 q_{11} = \lambda_2\{(\theta_{22}+\omega_{22})q_{11} - 2\omega_{12}q_{12} - 2\theta_{12}q_{21}\},$$

$$\delta_2 q_{12} = \lambda_2\{(\theta_{22}-\omega_{22})q_{12} + 2\theta_{12}q_{22}\},$$

$$\delta_2 q_{21} = -\lambda_2\{(\theta_{22}-\omega_{22})q_{21} + 2\omega_{12}q_{22}\},$$

$$\delta_2 q_{22} = -\lambda_2(\theta_{22}+\omega_{22})q_{22}, \tag{7.2.12}$$

$$\delta_3 q_{11} = -\lambda_3(\omega_{11}q_{12} + \theta_{11}q_{21}),$$

$$\delta_3 q_{12} = -\lambda_3(\omega_{22}q_{11} + \theta_{11}q_{22}),$$

$$\delta_3 q_{21} = -\lambda_3(\theta_{22}q_{11} + \omega_{11}q_{22}),$$

$$\delta_3 q_{22} = -\lambda_3(\theta_{22}q_{12} + \omega_{22}q_{21}), \tag{7.2.13}$$

It is now clear that the transformations $K_2(\lambda_2)$ and $K_3(\lambda_3)$ are independent of $K_1(\lambda_1)$. The actions of K_1, K_2 and K_3 on Θ and Ω are also obtained. Then it is possible to apply K_1, K_2 and K_3 to Q successively. This suggests that there exists an infinite-dimensional transformation group which is written formally as

$$\{SL(2,\mathbb{C})_G \times SL(2,\mathbb{C})_H \times SL(2,\mathbb{C})_G\} \times \{\quad\} \times \ldots$$

A similar transformation group called the Geroch group has been known for the gravitational fields. The Geroch group is a powerful tool to generate stationary axially symmetric vacuum solutions. In this sense, the above Geroch-like group is important for the study of gauge fields. Forgács, Horváth and Palla [4] recently have pointed out the existence of a similar infinite-dimensional group without giving explicite representations.

7.3. Nonlocal symmetries

An infinite number of nonlocal symmetries emerges in consequence of the infinite-dimensional transformation group discussed in Section 7.2. Let us recall the Zakharov-Shabat system (6.2.6) for the self-dual equations:

$$(\partial_{\bar{z}} + \zeta\partial_y)Y(\zeta) = \partial_{\bar{z}}P\cdot P^{-1}\,Y(\zeta),$$

$$(\partial_{\bar{y}} - \zeta\partial_z)Y(\zeta) = \partial_{\bar{y}}P\cdot P^{-1}\,Y(\zeta), \tag{7.3.1}$$

where $Y(\zeta)$ is a $n\times n$ matrix function of $(y, \bar{y}, z, \bar{z})$, analytic near $\zeta = 0$. Expanding $Y(\zeta)$ into a power series

$$Y(\zeta) = \sum_{n=0}^{\infty} Y_n\zeta^n, \tag{7.3.2}$$

we have the recursion relations

$$\partial_{\bar{z}}Y_{n+1} + \partial_y Y_n = \partial_{\bar{z}}P\cdot P^{-1}\,Y_{n+1},$$

$$\partial_{\bar{y}} Y_{n+1} - \partial_z Y_n = \partial_{\bar{y}} P \cdot P^{-1} Y_{n+1} \tag{7.3.3}$$

with $Y_0 = P$. Each Y_n gives a nonlocal conservation law

$$\partial_y (\partial_{\bar{y}} Y_n - \partial_{\bar{y}} P \cdot P^{-1} Y_n)$$
$$+ \partial_z (\partial_{\bar{z}} Y_n - \partial_{\bar{z}} P \cdot P^{-1} Y_n) = 0. \tag{7.3.4}$$

Pohlmeyer [11] has derived slightly different nonlocal conservation laws.

Now we define an $n \times n$ matrix $S(\zeta)$ by

$$S(\zeta) = Y(\zeta)V(\zeta)Y(\zeta)^{-1}, \tag{7.3.5}$$

where $V(\zeta)$ is assumed to satisfy $D_k V(\zeta) = 0$, $k = 1, 2$ with D_k defined by (6.2.5), and $V(0)$ is very close to zero. Let S_n be the coefficients of the Laurent expansion of $S(\zeta)$:

$$S(\zeta) = \sum_{n=-\infty}^{\infty} S_n \zeta^n. \tag{7.3.6}$$

These coefficients give nonlocal symmetries of the self-dual equations. To show this we consider the infinitesimal transformation

$$\Lambda: P \to P + \Lambda P, \tag{7.3.7}$$

where $\Lambda = \Lambda(y, \bar{y}, z, \bar{z})$ is an $n \times n$ matrix close to zero. We prove

Lemma 7.3.1. *If* Λ *satisfies the equation*

$$\Box\Lambda = [\partial_{\bar{y}} P\cdot P^{-1}, \partial_y\Lambda] + [\partial_{\bar{z}} P\cdot P^{-1}, \partial_z\Lambda], \qquad (7.3.8)$$

then Λ *gives a symmetru of the self-dual equations.*

The proof of this lemma is given by the same way as in Lemma 4.3.1. We have the following proposition.

Proposition 7.3.2. *Every* S_n *gives nonlocal symmetries of the self-dual equations.*

Proof. From (7.3.1) and (7.3.5) it follows that

$$(\partial_{\bar{z}} + \zeta\partial_y)S(\zeta) = [\partial_{\bar{z}} P\cdot P^{-1}, S(\zeta)],$$

$$(\partial_{\bar{y}} - \zeta\partial_z)S(\zeta) = [\partial_{\bar{y}} P\cdot P^{-1}, S(\zeta)]. \qquad (7.3.10)$$

Then from (7.3.6) we have

$$\partial_{\bar{z}} S_{n+1} + \partial_y S_n = [\partial_{\bar{z}} P\cdot P^{-1}, S_{n+1}],$$

$$\partial_{\bar{y}} S_{n+1} - \partial_z S_n = [\partial_{\bar{y}} P\cdot P^{-1}, S_{n+1}] \qquad (7.3.11)$$

with $S_0 = PV(0)P^{-1}$. The integrability condition with respect to S_n gives (7.3.8) of Lemma 7.3.1 for $\Lambda = S_n$. This proves the proposition. $\Box$

Remark 7.3.3. The generating function $S(\zeta)$ of symmetries is related to $H(\zeta)$ which appeared in the Riemann-Hilbert problem (6.2.8) by

$$S(\zeta) = H(\zeta) - 1, \tag{7.3.12}$$

if $S(\zeta)$ is analytic near $\zeta = 0$.

Hence, it is also possible to give a systematic treatment of the nonlocal symmetries by the use of infinitesimal Riemann-Hilbert transformations. We discuss this problem in the subsequent section.

7.4. Symmetry algebra

Hauser and Ernst [6] have exponentiated all of the Kinnersley-Chitre infinitesimal transformations of the Geroch group by means of the Riemann-Hilbert problem. Following their method, recently Ueno [12] has shown that the affine Lie algebras $su(n) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$, $so(n) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ act on solutions of $SU(n)$, $SO(n)$ chiral field equations, respectively.

In this section, we consider the infinitesimal Riemann-Hilbert transformations to construct the symmetry algebra of the $GL(n, \mathbb{C})$ self-dual equations. We have already proved Treorem 6.2.2 about transformations on solutions. Let us put the Riemann-Hilbert problem (6.2.8) and (6.2.9) into the Fredholm integral equation (see [10])

$$Y'(\zeta) - (Y' \circ K)(\zeta) = Y(\zeta)u(\zeta)^{-1}, \tag{7.4.1}$$

where the integral operator K is defined by

$$(F \circ K)(\zeta) = \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta' - \zeta} F(\zeta')$$

$$\times\{Y(\zeta')^{-1}Y(\zeta) - u(\zeta')Y(\zeta')^{-1}Y(\zeta)u(\zeta)^{-1}\}, \qquad \zeta \in C.$$

If $u(\zeta)$ is very close to unit matrix, the solution of (7.4.1) actually gives the solution of the Riemann-Hilbert problem, and consequently yields the Riemann-Hilbert transformation: $Y \to Y'$. Let us consider an infinitesimal transformation defined through (7.4.1). Assume that $u(\zeta)$ is of the form

$$u(\zeta) = \exp\, v(\zeta), \tag{7.4.2}$$

where $v(\zeta)$ is very close to zero. Then, solving (7.4.1) by the Neumann expansion, we see that $Y'(\zeta)$ equals approximately

$$Y'(\zeta) \sim Y(\zeta)u(\zeta)^{-1} + (Yu^{-1} \circ K)(\zeta)$$

$$\sim Y(\zeta) - Y(\zeta)v(\zeta) - \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta' - \zeta}$$

$$\times\{Y(\zeta)v(\zeta) - Y(\zeta')v(\zeta')Y(\zeta')^{-1}Y(\zeta)\}, \qquad \zeta \in C.$$

Here we neglect the terms of higher order in $v(\zeta)$. Since the integrand is analytic at $\zeta = \zeta'$, the variable ζ can be analytically continued into C_+. It should be noted that

$$\int_C \frac{d\zeta'}{\zeta' - \zeta} Y(\zeta)v(\zeta) = \int_C \frac{d\zeta'}{\zeta' - \zeta} Y(\zeta')v(\zeta'), \qquad \zeta \in C_+.$$

Then we obtain

$$Y'(\zeta) \sim Y(\zeta) - \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta' - \zeta} Y(\zeta')v(\zeta')$$

$$+ \frac{1}{2\pi i}\int_C d\zeta' Y(\zeta')v(\zeta')G(\zeta, \zeta'), \qquad (7.4.3)$$

for $\zeta \in C_+$, where $G(\zeta, \zeta')$ is difined by

$$G(\zeta, \zeta') = \frac{1}{\zeta - \zeta'}\{Y(\zeta')^{-1}Y(\zeta) - 1\}.$$

By the superscript n, m etc. in parenthsis we indicate the coefficients of the Taylor expansion; for example,

$$Y(\zeta) = \sum_{n=0}^{\infty} Y^{(n)}\zeta^n, \qquad G(\zeta, \zeta') = \sum_{m,n=0}^{\infty} G^{(m,n)}\zeta'^m\zeta^n.$$

From (7.4.3), we see that the coefficients $Y^{(n)}$ are infinitesimally transformed by the Riemann-Hilbert transformation $Y(\zeta) \to Y'(\zeta)$ as follows:

$$Y^{(n)} \to Y'^{(n)} \sim Y^{(n)} - \sum_{p=0}^{\infty} Y^{(p)} v^{(p-n)}$$

$$+ \sum_{p,q=0}^{\infty} Y^{(p)} v^{(p+q+1)} G^{(q,n)}, \qquad (7.4.4)$$

$$v^{(p)} = \frac{1}{2\pi i} \int_C d\zeta' v(\zeta')\zeta'^{p-1}. \qquad (7.4.5)$$

We call (7.4.4) *the infinitesimal Riemann-Hilbert transformation*. Furthermore we introduce

$$\hat{Y}(\zeta) = Y(0)^{-1}Y(\zeta) = P^{-1}Y(\zeta) = \sum_{n=0}^{\infty} \hat{Y}^{(n)}\zeta^n,$$

$$\hat{G}(\zeta, \zeta') = \frac{1}{\zeta' - \zeta}\{\zeta' - \zeta\hat{Y}(\zeta')^{-1}\hat{Y}(\zeta)\}$$

$$= \sum_{m,n=0}^{\infty} \hat{G}^{(m,n)}\zeta'^m\zeta^n. \qquad (7.4.6)$$

The $\hat{Y}^{(n)}$ and $\hat{G}^{(m,n)}$ are analogous to $E^{(n)}$ and $N^{(m,n)}$ defined by (5.1.13), and may be called potentials. The following lemma is crucial in our procedure.

Lemma 7.4.1. *The potentials* $\hat{G}^{(0,n)}$ *infinitesimally transform under the Riemann-Hilbert transformation according to*

$$\hat{G}^{(0,n)} \to \hat{G}^{(0,n)} - \sum_{p=0}^{\infty} \hat{G}^{(0,p)} v^{(p-n)}$$

$$+ \sum_{p,q=0}^{\infty} \hat{G}^{(0,p)} v^{(p+q)} \hat{G}^{(q,n)}. \tag{7.4.7}$$

Proof. From (7.4.4) we see that $\hat{Y}^{(n)}$ transform as follows:

$$\hat{Y}^{(n)} \to Y'(0)^{-1} Y'^{(n)}$$

$$\sim \{1 + \sum_{p=0}^{\infty} \hat{Y}^{(p)} v^{(p)} - \sum_{p,q=0}^{\infty} \hat{Y}^{(p)} v^{(p+q+1)} G^{(q,n)}\}$$

$$\times \{\hat{Y}^{(n)} - \sum_{p=0}^{\infty} \hat{Y}^{(p)} v^{(p-n)} + \sum_{p,q=0}^{\infty} \hat{Y}^{(p)} v^{(p+q+1)} G^{(q,n)}\}.$$

Neglecting the terms of higher order in $v(\zeta)$, we have

$$\hat{Y}^{(n)} \to \hat{Y}^{(n)} - \sum_{p=0}^{\infty} \hat{Y}^{(p)} v^{(p-n)} + \sum_{p=0}^{\infty} \hat{Y}^{(p)} v^{(p)} \hat{Y}^{(n)}$$

$$+ \sum_{p,q=0}^{\infty} \hat{Y}^{(p)} v^{(p+q+1)} \{G^{(q,n)} - G^{(q,0)} \hat{Y}^{(n)}\}.$$

On the other hand, by the definition (7.4.6), we obtain

$$\hat{G}^{(0,n)} = \hat{Y}^{(n)},$$

$$\hat{G}^{(m+1,n)} = G^{(m,n)} - G^{(m,0)} \hat{Y}^{(n)}, \tag{7.4.8}$$

for $m, n \geq 0$. Then the infinitesimal transformations read

$$\hat{G}^{(0,n)} \to \hat{G}^{(0,n)} - \sum_{p=0}^{\infty} \hat{G}^{(0,p)} v^{(p-n)} + \sum_{p=0}^{\infty} \hat{G}^{(0,p)} v^{(p)} \hat{G}^{(0,n)}$$

$$+ \sum_{p,q=0}^{\infty} \hat{G}^{(0,p)} v^{(p+q+1)} \hat{G}^{(q+1,n)}.$$

This proves Equation (7.4.7). □

Let us assume that in a certain annular domain $v(\zeta)$ is expanded into Larent series such that

$$v(\zeta) = \sum_{k=-\infty}^{\infty} \sum_{k_1,k_2=0}^{\infty} v^{(k,k_1,k_2)} w^k w_1^{k_1} w_2^{k_2} \qquad (7.4.9)$$

where $v^{(k,k_1,k_2)}$ are constant $gl(n, \mathbb{C})$-valued matrices (see (6.2.12) for w, w_1 and w_2). This assumption consists with the requirement (6.2.11). Let us consider the infinitesimal transformations induced by $v^{(k,k_1,k_2)}$. For simplicity, first we set

$$v(\zeta) = v^{(k)} w^k. \qquad (7.4.10)$$

From Lemma 7.4.1 and (7.4.5) with (7.4.10), we obtain

$$\hat{G}^{(0,n)} \to \hat{G}^{(0,n)} - \hat{G}^{(0,n+k)} v^{(k)} + v^{(k)} \hat{G}^{(k,n)}$$
$$+ \sum_{j=1}^{k} \hat{G}^{(0,j)} v^{(k)} \hat{G}^{(k-j,n)}, \qquad (7.4.11)$$

$$\hat{G}^{(0,n)} \to \hat{G}^{(0,n)} - \hat{G}^{(0,n+l)} v^{(l)}, \qquad (7.4.12)$$

for $k \geq 0$, $l \leq -1$, $n \geq 1$. It can be shown that the transformation (7.4.12) corresponds to the gauge transformation (7.4.8).

We proceed to consider the transformation of potentials $\{G^{(m,n)}\}$. Let us introduce the potentials $G^{(m,n)}$

with negative indices by

$$\hat{G}^{(l,-l)} = -\hat{G}^{(-l,l)} = 1, \qquad (7.4.13)$$

for $l \geq 1$, and $\hat{G}^{(m,n)} = 0$ for other negative indices. We have

Proposition 7.4.2. *The infinitesimal Riemann-Hilbert transformation induced by* $v(\zeta) = v^{(k)}w^k$ *acts on the manifold of potentials* $\{\hat{G}^{(m,n)}\}$ *as follows:*

$$v^{(k)}w^k:\ \hat{G}^{(m,n)} \to \hat{G}^{(m,n)} - \hat{G}^{(m,n+k)}v^{(k)} + v^{(k)}\hat{G}^{(m+k,n)}$$
$$+ \sum_{j=1}^{k} \hat{G}^{(m,j)}v^{(k)}\hat{G}^{(k-j,n)}, \qquad (7.4.14)$$

for $k \geq 0$, $m \geq 0$, $n \geq 1$.

Proof. The proof is carried out by induction. When $m = 0$, the claim of the proposition is true for $n \geq 1$ from (7.4.11). We note the recursion relation

$$\hat{G}^{(m,n)} = \hat{G}^{(m-1,n+1)} - \hat{G}^{(m-1,1)}\hat{G}^{(0,n)}. \qquad (7.4.15)$$

Suppose that (7.4.14) holds for $n \geq 1$ when $m = l-1 \geq 1$. Then the recursion relation and the assumption of induction give

$$v^{(k)}w^k:\ \hat{G}^{(l,n)} \to \hat{G}^{(l,n)} + v^{(k)}\hat{G}^{(l+k,n)} - \hat{G}^{(l,n+k)}v^{(k)}$$

$$- \hat{G}^{(l-1,1)} v^{(k)} \hat{G}^{(k,n)} + \hat{G}^{(l-1,k+1)} v^{(k)} \hat{G}^{(0,n)}$$

$$+ \sum_{j=1}^{k} \hat{G}^{(l-1,j)} v^{(k)} \hat{G}^{(k-j,n+1)}$$

$$- \sum_{j=1}^{k} \hat{G}^{(l-1,1)} \hat{G}^{(0,j)} v^{(k)} \hat{G}^{(k-j,n)}$$

$$- \sum_{j=1}^{k} \hat{G}^{(l-1,j)} v^{(k)} \hat{G}^{(k-j,1)} \hat{G}^{(0,n)} .$$

Here we neglect the higher order terms in $v^{(k)}$. We again use the recursion relation (7.4.15) with $m = l$ and $n = k$. This shows (7.4.14) with $m = l$. This completes the proof.□

The gauge transformations for $\hat{G}^{(m,n)}$ are given by the following proposition. We have obtained those for $\hat{G}^{(0,n)}$ in (7.4.12).

Proposition 7.4.3. *The infinitesimal Riemann-Hilbert transformation induced by* $v(\zeta) = v^{(l)} w^{l}$, $l \leq -1$, *acts on the manifold of potentials* $\{G^{(m,n)}\}$ *as follows:*

$$v^{(l)} w^{l} : \hat{G}^{(m,n)} \to \hat{G}^{(m,n)} - \hat{G}^{(m,n+l)} v^{(l)} + v^{(l)} \hat{G}^{(m+l,n)}$$
$$+ \sum_{j=0}^{-l-1} \delta_{m,j} v^{(l)} \delta_{-l-j,n} . \qquad (7.4.16)$$

Proof. By (7.4.13), Equation (7.4.16) is reduced to (7.4.12) when $m = 0$. The proof can be continued in the same way as in Proposition 7.4.2. □

In general we can naturally regard an infinitesimal transformation induced by

$$v(\zeta) = v^{(k,k_1,k_2)} w^k w_1^{k_1} w_2^{k_2} \tag{7.4.17}$$

as a linear combination of (7.4.14) and (7.4.16). To see this, we expand $w_1^{k_1}$ and $w_2^{k_2}$ in (7.4.17) to get

$$v(\zeta) = v^{(k,k_1,k_2)} w^k \{ \sum_{j_1=0}^{k_1} \binom{k_1}{j_1} \bar{y}^{j_1} (wz)^{k_1-j_2} \}$$

$$\times \{ \sum_{j_2=0}^{k_2} \binom{k_2}{j_2} \bar{z}^{j_2} (-wy)^{k_2-j_2} \}. \tag{7.4.18}$$

The action associated to (7.4.17) is given in terms of (7.4.18). Let us define

$$g = \text{span of } \{ v^{(k,k_1,k_2)} w^k w_1^{k_1} w_2^{k_2} \,\big|\, v^{(k,k_1,k_2)} \in gl(n, \mathbb{C}),\ k_1, k_2 \geq 0 \}.$$

Then we find g to be equipped with the structure of a graded Lie algebra. We have the main theorem.

Theorem 7.4.4. *The generators* $v^{(k,k_1,k_2)} w^k w_1^{k_1} w_2^{k_2}$ *and* $v^{(l,l_1,l_2)} w^l w_1^{l_1} w_2^{l_2}$ *satisfy the commutation relation*

$$[\![v^{(k,k_1,k_2)} w^k w_1^{k_1} w_2^{k_2},\ v^{(l,l_1,l_2)} w^l w_1^{l_1} w_2^{l_2}]\!]$$

$$= [v^{(k,k_1,k_2)},\ v^{(l,l_1,l_2)}]\ w^{k+l} w_1^{k_1+l_1} w_2^{k_2+l_2}, \tag{7.4.19}$$

where the bracket $[\![\ ,\ ]\!]$ *is defined for the infinitesimal transformations induced by* (7.4.17), *so that* (7.4.19) *is the equation for infinitesimal transformations. The bracket* $[\ ,\ ]$ *is the usual one of the Lie algebra* $gl(n, \mathbb{C})$.

Proof. For simplicity, we abbreviate $v^{(k,k_1,k_2)}$ and $v^{(l,l_1,l_2)}$ to v and v', respectively. The proof is broken up into four cases. First we prove (7.4.19) for the case $k, l \geq 0$. By making use of Proposition 7.4.2 for any nonnegative integers k and l, we have under $[\![vw^k, v'w^l]\!]$,

$$\begin{aligned}
\hat{G}^{(m,n)} \to \hat{G}^{(m,n)} &+ [v, v']\hat{G}^{(m+k+l,n)} - \hat{G}^{(m,n+k+l)}[v, v'] \\
&- \sum_{j=1}^{k} \hat{G}^{(m,l+j)} v'v\hat{G}^{(k-j,n)} \\
&- \sum_{j=1}^{k} \hat{G}^{(m,j)} vv'\hat{G}^{(k+l-j,n)} \\
&- \sum_{j=1} \hat{G}^{(m,k+j)} vv'\hat{G}^{(l-j,n)} \\
&- \sum_{j=1} \hat{G}^{(m,j)} v'v\hat{G}^{(k+l-j,n)} \\
= \hat{G}^{(m,n)} &+ [v, v']\hat{G}^{(m+k+l,n)} - \hat{G}^{(m,n+k+l)}[v, v'] \\
&+ \sum_{j=1}^{k+l} \hat{G}^{(m,j)}[v, v']\hat{G}^{(k+l-j,n)}.
\end{aligned}$$

This implies $[\![vw^k, v'w^l]\!] = [v, v']w^{k+l}$. From (7.4.18), we see that (7.4.19) holds for $k, l \geq 0$. The second case, $k, l < 0$, can be proved in the same way as above by Proposition 7.4.3.

Next, we proceed to the third case $k \geq -l > 0$. The proof is rather complicated than those of the previous two cases. Neglecting higher order terms more than third in v and v', we obtain under $[\![vw^k, v'w^l]\!]$,

$$\begin{aligned}\hat{G}^{(m,n)} \to \hat{G}^{(m,n)} &+ [v, v']\hat{G}^{(m+k+l,n)} - \hat{G}^{(m,n+k+l)}[v, v'] \\ &+ \sum_{j=1}^{k+l} \hat{G}^{(m,j)}vv'\hat{G}^{(k+l-j,n)} \\ &- \sum_{j=1}^{k+l} \hat{G}^{(m,j)}v'v\hat{G}^{(k+l-j,n)},\end{aligned}$$

where we have used $\delta_{m+k,j} = \delta_{-l-j,n+k} = 0$ for $k-1 \geq -l-1 \geq j \geq 0$, $m \geq 0$ and $n \geq 1$. Hence we see (7.4.19) for $k \geq -l > 0$. The last case, $-l > k > 0$ can be proved by the same way. This completes the proof. $\square$

As a corollary of Theorem 7.4.4, we have

Corollary 7.4.5. *The algebra* g *which the infinitesimal Riemann-Hilbert transformations form is isomorphic to*

$$gl(n, \mathbb{C}) \otimes \mathbb{C}[w, w^{-1}, w_1, w_2]. \tag{7.4.20}$$

The infinite-dimensional Lie algebra (7.4.20) is more complicated than $sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ for the stationary axially symmetric vacuum gravitational fields [8] and $su(n) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ for the $SU(n)$ chiral fields [12]. This is due to the fact that the self-dual equations are defined on the four-dimensional space, while the others are defined on the two-dimensional space.

The Lie algebras acting on G chiral fields are independently found by Dolan [2]. She derives the subalgebra $g \otimes \mathbb{C}[\zeta]$ by the method of variation. Recently along this line of thought, Chau, Ge and Wu [1] present the Lie algebra $gl(n, \mathbb{C}) \otimes \mathbb{C}[\zeta]$ for the self-dual equations. The link between the infinitesimal Riemann-Hilbert transformations and the infinitesimal transformations of [1, 2] is discussed in the subsequent chapter.

CHAPTER VIII

NOETHER TRANSFORMATIONS FOR NONLINEAR FIELDS

The discovery of conserved currents for the chiral field equations (Section 2.4) suggests that nontrivial symmetries should exist. In fact, by a similar calculation performed in Section 7.3, a set of symmetries which make the chiral field equations invariant can be derived from the inverse scattering formula. These symmetries are often referred to as *the symmetry transformations of the equations of motion* [13]. What we will study in this chapter is concerned with so-called *Noether transformations* which do not refer to the equations of motion. By definition [3] the Noether transformation makes a change in the Lagrangian density by adding a total divergence term and gives weak continuity equation, the definition of which will be given later, or a conserved current as a Noether current. We will discuss the Noether transformation for nonlinear fields.

Two Noether transformations called *the hidden symmetries* for the chiral fields were discovered by Dolan and Roos [7]. These prove soon to be the first two of a sequence of hidden symmetries which are derived by using one of the inverse scattering formula [5,10]. These Noether transformations reduce to the symmetry transformations when the fields satisfy the equations of motion.

The first purpose of this chapter is to propose a method for generating Noether transformations for nonlinear fields such that even for the solution fields the resulting Noether transformations are not necessarily the symmetry transformations.

We have discussed in Section 7.4 the infinitesimal Riemann-Hilbert transformations for the self-dual Yang-Mills equations. These form an infinite-dimensional Lie algebra. A similar Lie algebra acting on the solution space of the chiral field equations, a symmetry algebra, has been given in [14]. On the other hand, Dolan [6] has pointed out the hidden symmetries of the chiral fields form a infinite-dimensional Lie algebra called the hidden symmetry algebra. Our second purpose is to study the relationship between the hidden symmetries and the infinitesimal Riemann-Hilbert transformations for the chiral fields.

In Section 8.1, we review Dolan's hidden symmetry and the Noether current in a refined form. In Section 8.2, a method for finding Noether transformations of nonlinear fields is proposed. Weak continuity equations and conserved quantities of the chiral fields and of the Yang-Mills-Higgs fields are derived explicitely. In Section 8.3, we show that Dolan's hidden symmetry is essentially deduced from the infinitesimal Riemann-Hilbert transformation. We also discuss the hidden symmetry algebra for the chiral fields*. Calculations are performed in the Euclidean space, but can be extended without difficulty to the Minkowski space.

8.1. Hidden symmetries and Noether currents for the chiral fields

Let us recall the Lagrangian density for the G invariant two-dimensional chiral fields (2.1.9) and the equations of motion (2.1.10). The inverse scattering formula (2.3.9) is expressed as

$$(\partial_1 - \zeta\partial_0)Y(\zeta) = \zeta A_0 Y(\zeta), \tag{8.1.1}$$

$$(\partial_0 + \zeta\partial_1)Y(\zeta) = -A_1 Y(\zeta), \tag{8.1.2}$$

where $A_\mu = g^{-1}\partial_\mu g$, $\mu = 0, 1$, and $g(x) \in G$. The compatibility condition of (8.1.1) and (8.1.2) gives (2.1.10). Expanding a fundamental solution matrix $Y(\zeta)$ near $\zeta = 0$, $Y(\zeta) = \sum_{n=0}^{\infty} Y_n \zeta^n$, we have the recursion relations

---

* Section 8.3 is part of the collaboration with Lecturer Kimio UENO.

$$\partial_\mu Y_n = -\varepsilon_{\mu\nu}(\partial_\nu + A_\nu)Y_{n-1}, \quad Y_0 = 1, \tag{8.1.3}$$

where $\varepsilon_{10} = 1$. Then we obtain the conserved currents

$$j_\mu^{(n)} = \partial_\mu Y_n + A_\mu Y_n, \quad j_\mu^{(1)} = A_\mu \tag{8.1.4}$$

for the chiral field equations (2.1.10). The proof is as follows: Using (8.1.3), we get

$$\partial_\mu j_\mu^{(n)} = \partial_\mu A_\mu \cdot Y_n - \varepsilon_{\mu\nu}(\partial_\mu + A_\mu)(\partial_\nu + A_\nu)Y_{n-1}.$$

The second term in the right-hand side vanishes from the definition of A_μ. Thus we see that $\partial_\mu j_\mu^{(n)} \overset{\circ}{=} 0$, where the notation $\overset{\circ}{=}$ indicates that the equality holds for all solutions of the equations of motion; $\partial_\mu A_\mu = 0$. We call $\partial_\mu j_\mu^{(n)} \overset{\circ}{=} 0$ *the weak continuity equations* (see [3]). The conserved currents $j_\mu^{(n)}$ were given by Brézin, Itzykson, Zinn-Justin and Zuber [2].

We set

$$S(\zeta) = Y(\zeta) v\, Y(\zeta)^{-1}, \tag{8.1.5}$$

where v is a constamt matrix in the Lie algebra g of G. From (8.1.1), the function $S(\zeta)$ satisfies

$$(\partial_1 - \zeta\partial_0)S(\zeta) = \zeta[A_0, S(\zeta)]. \tag{8.1.6}$$

The Taylor expansion $S(\zeta) = \sum_{n=0}^{\infty} \lambda_n \zeta^n$ gives the recursion relations

$$\partial_1\lambda_n = \partial_0\lambda_{n-1} + [A_0, \lambda_{n-1}], \qquad \lambda_0 = v, \tag{8.1.7}$$

or equivalently

$$\lambda_n(x^0, x^1) = \int_{-\infty}^{x^1} dy\{\partial_0\lambda_{n-1}(x^0, y)$$

$$+ [A_0(x^0, y), \lambda_{n-1}(x^0, y)]\}. \tag{8.1.8}$$

Dolan and Roos [7] proposed the infinitesimal transformations defined by

$$g \to g + \delta_v^{(n)} g, \qquad \delta_v^{(n)} g = -g\lambda_n. \tag{8.1.9}$$

The transformation (8.1.9) have the following remarkable properties. First, each of them transforms the Lagrangian density by adding a total divergence term without referring to the equations of motion, so that (8.1.9) are Noether transformations for the chiral fields. In fact, without using (8.1.2), we can show that under the infinitesimal transformation $\delta_v g = -gS(\zeta)$ the change in the Lagrangian density is of the form

$$\delta L = \partial_\mu K_\mu, \tag{8.1.10}$$

$$K_\mu = \varepsilon_{\mu\nu}\mathrm{Tr}\{\zeta A_\nu S(\zeta) + (\zeta + \zeta^{-1})Y(\zeta)^{-1}\partial_\nu Y(\zeta)v\}.$$

Secondly, δL results in Noether currents which are expressed as

$$j_\mu = \frac{\delta L}{\delta(\partial_\mu g)} \delta_v g - K_\mu . \tag{8.1.11}$$

For the solutions of the equations of motion, these currents reduce to (8.1.4). The third is a group property discussed in [6]. The generators introduced by

$$M^{(n)}(v) = -\int d^2y \ \delta_v^{(n)} g(y) \frac{\delta}{\delta g(y)} \tag{8.1.12}$$

satisfy the commutation relations

$$[\![M^{(n)}(v), M^{(m)}(v')]\!] = M^{(n+m)}([v, v']), \tag{8.1.13}$$

for any $v, v' \in g$ and $n, m \geq 0$. The algebra formed by (8.1.12) is isomorphic to the subalgebra $g \otimes \mathbb{C}[\zeta]$ of the affine Lie algebra $g \otimes \mathbb{C}[\zeta, \zeta^{-1}]$. The Noether transformations (8.1.9) are frequnently referred to as the hidden symmetries.

8.2. Method for finding Noether transformations

In this section, we propose a method for finding Noether transformations which is out of the work of Dolan and Roos. We start by discussing the chiral fields. Consider a gauge transformation

$$g \to gU, \quad A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U, \tag{8.2.1}$$

where $U = U(x) \in G$. We restrict ourselves to the infinitesimal transformation induced by V;

$$U = \exp V, \quad V = V(x) \in g. \tag{8.2.2}$$

The Lagrangian density transforms as $L \to L + \delta L$. To first order we have

$$\delta L = \mathrm{Tr}\{\partial_\mu A_\mu \cdot V - \partial_\mu(A_\mu V)\}. \tag{8.2.3}$$

We assume that there is a matrix $W = W(x)$ which is related to $V(x)$ by the equations

$$\partial_\mu V = -\varepsilon_{\mu\nu}(\partial_\nu + A_\nu)W. \tag{8.2.4}$$

The integrability condition leads to

$$\partial_\mu{}^2 V + A_\mu \partial_\mu V = 0, \tag{8.2.5}$$

where we have used the identity $\varepsilon_{\mu\nu}(\partial_\mu A_\nu + A_\mu A_\nu) = 0$. Suppose that the transformations (8.2.1) and (8.2.2) are given by $V(x)$ which satisfies (8.2.5). Then the Lagrangian density is invariant up to the divergence

$$\delta L = -\mathrm{Tr}(A_\mu \partial_\mu V) = -\mathrm{Tr}\{\varepsilon_{\mu\nu}\partial_\mu(A_\nu W)\}. \tag{8.2.6}$$

As Equations (8.2.3) and (8.2.6) coincide, we have

Proposition 8.2.1. *Let* J_μ *be*

$$J_\mu = A_\mu V - \varepsilon_{\mu\nu} A_\nu W. \tag{8.2.7}$$

Then J_μ *satisfies the weak continuity equation*

$$\partial_\mu J_\mu \overset{\circ}{=} 0. \tag{8.2.8}$$

Proof. By direct calculation, we have

$$\partial_\mu J_\mu = \partial_\mu A_\mu \cdot V - \varepsilon_{\mu\nu}(\partial_\mu A \;\; + A_\mu A_\nu)W$$
$$+ A_\mu\{\partial_\mu V + \varepsilon_{\mu\nu}(\partial_\nu + A_\nu)W\}.$$

From (8.2.4) with the equations of motion, it follows that $\partial_\mu J_\mu \overset{\circ}{=} 0$. This proves the proposition. □

We now come to the conclusion that the function $V(x)$ satisfing (8.2.5) yields a Noether transformation. In deriving a conservation laws from the weak continuity equation, it is necessary to make an assumption about the behavior of the field $g(x^0, x^1)$ at spatial infinity. This assumption is of a physical nature and should be satisfied by solutions of the equations of motion. We have

Proposition 8.2.2. *Suppose* $\lim_{|x^1| \to \infty} g(x^0, x^1) = 1$. *Then*

$$\int_{-\infty}^{\infty} dy\{A_0(x^0, y)V(x^0, y) - A_1(x^0, y)W(x^0, y)\} \qquad (8.2.9)$$

is a constant of motion.

The Proof is performed by integrating (8.2.8) over $\mathbb{R}$ and using the boundary condition.

The condition that the infinitesimal transformation (8.2.1) with (8.2.2) be the symmetry transformation is expressed as

$$\partial_\mu{}^2 V + [A_\mu, \partial_\mu V] = 0. \qquad (8.2.10)$$

Combining (8.2.5) with (8.2.10), we obtain

$$\partial_\mu V \cdot A_\mu = 0. \qquad (8.2.11)$$

This is an additional constraint to be imposed on the Noether transformation if it is to lead to a symmetry of the equations of motion.

We now show the relationship between the Noether currents (8.2.7) and the known conserved currents (8.1.4). For this purpose, we identify $V(x)$ and $W(x)$ with Y_n and Y_{n-1} in (8.1.3), respectively. Then the currents (8.2.7) become

$$J_\mu = \varepsilon_{\mu\nu}\partial_\nu Y_{n+1} + \varepsilon_{\mu\nu}\partial_\nu Y_{n-1}.$$

Since $j_\mu^{(n)} = \varepsilon_{\mu\nu}\partial_\nu Y_{n+1}$, we obtain

$$J_\mu = j_\mu^{(n)} + j_\mu^{(n-2)}. \tag{8.2.12}$$

The Noether currents (8.2.7) are thus expressed as a linear combination of the known conserved currents.

Next, we consider the Noether transformation of the Yang-Mills-Higgs fields. The Lagrangian density takes the form [9]

$$L = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) - \mathrm{Tr}(D_\mu\phi D_\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - 1)^2, \tag{8.2.13}$$

where ϕ is a Higgs field valued in g and $\lambda \geq 0$. The variation of the action in no-Higgs self-interraction limit $\lambda \to 0$ yields the equations of motion

$$D_\mu F_{\mu\nu} + [D_\mu\phi, \phi] = 0, \quad D_\mu D_\mu\phi = 0. \tag{8.2.14}$$

If the Higgs field vanishes, Equations (8.2.14) reduce to the Yang-Mills equations (6.1.6) admitting the finite action solutions called the instantons. In static case, finite energy solutions called the monopoles of (8.2.14) are found [12, 15] by solving

$$F_{\mu\nu} = \varepsilon_{\mu\nu\sigma}D_\sigma\phi \tag{8.2.15}$$

which are referred to as the Bogomol'nyi equations [1].

Let ξ_μ and η be infinitesimal changes in B_μ and

ϕ, respectively:

$$B_\mu \to B_\mu + \xi_\mu, \qquad \phi \to \phi + \eta. \tag{8.2.16}$$

This transformation makes a change in the Lagrangian density in the limit $\lambda \to 0$ by

$$\begin{aligned}\delta L &= -\,\mathrm{Tr}\{F_{\mu\nu}(D_\mu\xi_\nu - D_\nu\xi_\mu) + 2D_\mu\phi(D_\mu\eta + [\xi_\mu,\ \phi])\} \\ &= 2\mathrm{Tr}\{(D_\mu F_{\mu\nu} + [D_\nu\phi,\ \phi])\xi_\nu + D_\mu D_\mu\phi\cdot\eta \\ &\qquad - \partial_\mu(F_{\mu\nu}\xi_\nu) - \partial_\mu(D_\mu\phi\cdot\eta)\}.\end{aligned} \tag{8.2.17}$$

We introduce the potentials κ_μ such that

$$D_\mu\xi_\nu - D_\nu\xi_\mu = \varepsilon_{\mu\nu\rho\sigma}D_\rho\kappa_\sigma \tag{8.2.18}$$

and impose the constraint

$$D_\mu\eta + [\xi_\mu,\ \phi] = 0 \tag{8.2.19}$$

on η. Then we have

$$\delta L = -\,\mathrm{Tr}\{\partial_\rho(\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}\kappa_\sigma)\}. \tag{8.2.20}$$

Here we have used the Bianchi identity $\varepsilon_{\mu\nu\rho\sigma}D_\rho F_{\mu\nu} = 0$. As (8.2.17) is equal to (8.2.20), we prove the following proposition.

Proposition 8.2.3. *Let* Z_μ *be*

$$Z_\mu = \mathrm{Tr}\{F_{\mu\nu}\cdot\xi_\nu + D_\mu\phi\cdot\eta - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}\kappa_\sigma\}. \tag{8.2.21}$$

Then Z_μ *satisfy the weak continuity equation*

$$\partial_\mu Z_\mu \overset{\circ}{=} 0 \tag{8.2.22}$$

for the Yang-Mills-Higgs fields.

Proof. The left-hand side of (8.2.22) is evaluated to be

$$\begin{aligned}\partial_\mu Z_\mu = \mathrm{Tr}\{&(D_\mu F_{\mu\nu} + [D_\nu\phi,\ \phi])\xi_\nu + D_\mu D_\nu\phi\cdot\eta \\ &- \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}D_\mu F_{\nu\rho}\kappa_\sigma + (D_\mu\eta + [\xi_\mu,\ \phi])D_\mu\phi \\ &+ F_{\mu\nu}(D_\mu\xi_\nu - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}D_\rho\kappa_\sigma)\}.\end{aligned}$$

Using the equations of motion (8.2.14), we obtain the weak continuity equation $\partial_\mu Z_\mu \overset{\circ}{=} 0$. □

An implication of this proposition is that every set $(\xi_\mu,\ \kappa_\mu,\ \eta)$ which solves (8.2.18) and (8.2.19) is associated with a Noether transformation of the Yang-Mills-Higgs fields. By identifying x^0 with the imaginary time, we obtain the conserved charge

$$Q = \int d^3x\ \mathrm{Tr}(F_{0\nu}\cdot\xi_\nu + D_0\phi\cdot\eta - \tfrac{1}{2}\varepsilon_{0\nu\rho\sigma}F_{\rho\sigma}\kappa_\nu). \tag{8.2.23}$$

It should be noted that for $\eta = 0$ this conserved charge reduces to that of the Yang-Mills fields given by Chodos [4]. By taking the dual of (8.2.18) we have

$$D_\mu \kappa_\nu - D_\nu \kappa_\mu = \varepsilon_{\mu\nu\rho\sigma} D_\rho \xi_\sigma, \tag{8.2.24}$$

so that for each $(\xi_\mu, \kappa_\mu, \eta)$ we get another set $(\kappa_\mu, \xi_\mu, \eta)$.

Suppose the infinitesimal transformation (8.2.16) be a symmetry transformation. To first order in ξ_μ and η we have from (8.2.14)

$$\begin{aligned} &D_\mu (D_\mu \xi_\nu - D_\nu \xi_\mu) + [\xi_\mu, F_{\mu\nu}] + [D_\nu \phi, \eta] \\ &\quad + [[\xi_\nu, \phi] + D_\nu \eta, \phi] = 0. \quad D_\mu D_\mu \eta = 0. \end{aligned} \tag{8.2.25}$$

If ξ_μ and η also satisfy the condition of Noether transformation (8.2.18) and (8.2.19), the first of (8.2.25) can be written as

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[F_{\mu\rho}, \kappa_\sigma] + [\xi_\mu, F_{\mu\nu}] + [D_\nu \phi, \eta] = 0. \tag{8.2.26}$$

Here we have used $\varepsilon_{\mu\nu\rho\sigma} D_\mu D_\rho \kappa_\sigma = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[F_{\mu\rho}, \kappa_\sigma]$. Hence the Noether transformation given by (8.2.18) and (8.2.19) does not necessarily lead to symmetry of the equations of motion.

8.3. Hidden symmetries and Riemann-Hilbert transformations

In this section, it is shown that an infinitesimal Riemann-Hilbert transformation gives a hidden symmetry of chiral fields. The hidden symmetry algebra proposed by Dolan [7] is identified with the algebra of the infinitesimal Riemann-Hilbert transformations.

First we put the hidden symmetry (8.1.9) into the language of the Riemann-Hilbert problem discussed in Sections 4.3 and 6.2. Let us start with a fundamental solution matrix $Y(\zeta)$ of (8.1.1) analytic near the origin $\zeta = 0$ together with $Y(0) = 1$ and $\det Y(\zeta) = 1$. Since the hidden symmetry (8.1.9) induces the infinitesimal transformation on the potential A_0,

$$A_0 \to A_0 - \delta_v^{(n)} A_0, \quad \delta_v^{(n)} A_0 = \partial_0 \lambda_n + [A_0, \lambda_n], \tag{8.3.1}$$

the fundamental solution $Y(\zeta)$ of (8.1.1) is subject to the infinitesimal transformation as

$$\begin{aligned} &Y(\zeta) \to Y(\zeta) - Z^{(n)}(\zeta) Y(\zeta), \\ &(\partial_1 - \zeta\partial_0) Z^{(n)}(\zeta) = \zeta[A_0, Z^{(n)}(\zeta)] + \zeta\delta_v^{(n)} A_0. \end{aligned} \tag{8.3.2}$$

In the following discussions we will show that the infinitesimal Riemann-Hilbert transformation, $Y \to Y'$, solves

Equations (8.3.1) and (8.3.2).

For simplicity, let the Lie group G in which the field $g(x)$ takes its value be $SL(N, \mathbb{C})$. Consider the Riemann-Hilbert problem of finding $X_{\pm}(\zeta)$ such that

$$X_{-}(\zeta) = X_{+}(\zeta)H(\zeta) \tag{8.3.3}$$

on a curve C, where $H(\zeta) = Y(\zeta)u(\zeta)Y(\zeta)^{-1}$ and $X_{+}(0) = 1$. Here C is a small circle with the center at $\zeta = 0$ such that $Y(\zeta)$ is analytic in $C \cup C_{+}$. The symbols C_{+} and C_{-} denote the inside and outside of C, respectively. The $SL(N, \mathbb{C})$-valued matrix $u(\zeta)$ is independent of x^0, x^1 and is analytic on C. We asuume that there is a unique pair of solution matrices $X_{\pm}(\zeta)$ to (8.3.3) such that $X_{+}(\zeta)$ (resp. $X_{-}(\zeta)$) is analytic and non-singular in $C \cup C_{+}$ (resp. $C \cup C_{-}$). As an application of the Riemann-Hilbert problem (8.3.3), we introduce a Riemann-Hilbert transformation. Define $Y'(\zeta)$ and A_0' by

$$Y'(\zeta) = \begin{cases} X_{+}(\zeta)Y(\zeta) & \text{in } C_{+}, \\ X_{-}(\zeta)Y(\zeta)u(\zeta)^{-1} & \text{in } C_{-}, \end{cases} \tag{8.3.4}$$

$$A_0' = A_0 + \partial_1 \dot{X}_{+}(0), \tag{8.3.5}$$

where the dot stands for the differentiation with respect to ζ. It follows that $Y'(\zeta)$ is analytic near $\zeta = 0$,

$Y'(0) = 1$ and $\det Y'(\zeta) = 1$. Then we have

Proposition 8.3.1. *The matrix* $Y'(\zeta)$ *is a fundamental solution of*

$$(\partial_1 - \zeta\partial_0)Y'(\zeta) = \zeta A_0' Y'(\zeta). \tag{8.3.6}$$

Proof. From (8.1.1) and (8.3.3), we derive

$$(\partial_1 - \zeta\partial_0)X_+ \cdot X_+^{-1} + \zeta X_+ A_0 X_+^{-1}$$
$$= (\partial_1 - \zeta\partial_0)X_- \cdot X_-^{-1} + \zeta X_- A_0 X_-^{-1}.$$

Hence the left-hand side can be analytically continued to the whole ζ-plane, which has a simple pole at $\zeta = \infty$. Then the left-hand side takes the form

$$(\partial_1 - \zeta\partial_0)X_+ \cdot X_+^{-1} + \zeta X_+ A_0 X_+^{-1} = P + \zeta Q,$$

linear in ζ at most. Substituting $\zeta = 0$ into the above equation, we see $P = 0$. To first order in ζ, $Q = A_0'$ $= A_0 + \partial_1 \dot{X}_+(0)$. Incidentally, from (8.1.1) and (8.3.4) it follows that

$$(\partial_1 - \zeta\partial_0)Y' = (\partial_1 - \zeta\partial_0)X_+ \cdot Y + \zeta X_+ A_0 Y.$$

Then the right-hand side of this reads $\zeta QY' = \zeta A_0' Y'$. This proves the proposition. □

Equations (8.3.4) are called the Riemann-Hilbert transformation for the chiral fields without referring to the equations of motion. We proceed along the line developed in Section 7.4. It is known that the problem (8.3.3) can be rewritten into the integral equation [11]

$$X_+(\zeta) - \frac{1}{2\pi i} \int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} X_+(\zeta)$$
$$\times\{1 - H(\zeta')H(\zeta)^{-1}\} = 1, \quad \zeta \in C_+. \qquad (8.3.7)$$

If $u(\zeta)$ is very close to the unit matrix, the unique solution of (8.3.7) actually gives that of (8.3.3). After [14], we consider the infinitesimal Riemann-Hilbert transformation induced by $v(\zeta)$:

$$u(\zeta) = \exp v(\zeta), \qquad (8.3.8)$$

where $v(\zeta)$ is an $sl(N, \mathbb{C})$ matrix and very close to 0. Solving (8.3.7) by means of the Neumann expansion, and neglecting higher order terms in $v(\zeta)$, we have

Proposition 8.3.2. *The solution matrix* $X_+(\zeta)$ *of* (8.3.7) *approximately equals*

$$X_+(\zeta) \sim 1 - Z(\zeta),$$

$$Z(\zeta) = \frac{1}{2\pi i} \int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} R(\zeta'), \quad \zeta \in C_+,$$

$$R(\zeta) = Y(\zeta)v(\zeta)Y(\zeta)^{-1}. \tag{8.3.9}$$

Proof. From the Neumann expansion of (8.3.7), it follows that

$$X_+(\zeta) \sim 1 + \frac{1}{2\pi i}\int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)}\{1 - H(\zeta')H(\zeta)^{-1}\}.$$

To first order in $v(\zeta)$

$$H(\zeta')H(\zeta)^{-1} \sim 1 + R(\zeta') - R(\zeta).$$

Noting that

$$\int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} = 0, \qquad \zeta \in C_+,$$

we obtain $X_+(\zeta) \sim 1 - Z(\zeta)$. This proves Proposition 8.3.2. □

Thus we have shown that the infinitesimal Riemann-Hilbert transformation takes the form

$$Y(\zeta) \to Y'(\zeta) \sim Y(\zeta) - Z(\zeta)Y(\zeta). \tag{8.3.10}$$

In view of (8.3.2), we prove the following propositions.

Proposition 8.3.3. *The infinitesimal Riemann-Hilbert transformation* (8.3.10) *induces a transformation*

$$A_0 \to A_0 - \{\partial_0\chi + [A_0, \chi]\},$$

$$= \frac{1}{2\pi i} \int_C \frac{d\zeta'}{\zeta'} R(\zeta') \tag{8.3.11}$$

on the potential A_0.

Proof. From (8.3.5) and (8.3.9), we see that A_0 is transformed to $A_0 - \partial_1 \dot{Z}(0)$ under the transformation (8.3.10). On the other hand, using

$$\dot{Z}(\zeta) = \frac{1}{2\pi i} \int_C \frac{d\zeta'}{(\zeta')^2} R(\zeta'),$$

$$\partial_1 Y(\zeta') \cdot Y(\zeta')^{-1} = \zeta'(\partial_0 Y(\zeta') \cdot Y(\zeta')^{-1} + A_0),$$

we write out $\partial_1 \dot{Z}(0)$ to be

$$\begin{aligned}\partial_1 \dot{Z}(0) &= \frac{1}{2\pi i} \int_C \frac{d\zeta'}{\zeta'} [\partial_0 Y(\zeta') \cdot Y(\zeta')^{-1}, R(\zeta')] \\ &\quad + \frac{1}{2\pi i} \int_C \frac{d\zeta'}{\zeta'} [A_0, R(\zeta')] \\ &= \frac{1}{2\pi i} \int_C \frac{d\zeta'}{\zeta'} (\partial_0 R(\zeta') + [A_0, R(\zeta')]).\end{aligned}$$

This completes the proof. □

Proposition 8.3.4. *The matrix* $Z(\zeta)$ *defined by* (8.3.9) *satisfies*

$$\begin{aligned}&(\partial_1 - \zeta\partial_0) Z(\zeta) \\ &\quad = \zeta[A_0, Z(\zeta)] + \zeta(\partial_0 \chi + [A_0, \chi]).\end{aligned} \tag{8.3.12}$$

Proof. From the proof of Proposition 8.3.3, we have

$$\partial_1 R(\zeta') = \zeta'(\partial_0 R(\zeta') + [A_0, R(\zeta')]). \qquad (8.3.13)$$

Then the left-hand side of (8.3.12) reads

$$\frac{1}{2\pi i}\int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} [(\partial_1 - \zeta'\partial_0)R(\zeta') + (\zeta' - \zeta)\partial_0 R(\zeta')]$$

$$= \frac{1}{2\pi i}\int_C \frac{\zeta d\zeta'}{\zeta' - \zeta} [A_0, R(\zeta')] + \frac{1}{2\pi i}\int_C \frac{\zeta d\zeta'}{\zeta'}\partial_0 R(\zeta'),$$

for $\zeta \in C_+$. Since

$$\frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta' - \zeta} R(\zeta') = \frac{1}{2\pi i}\int_C \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} R(\zeta')$$

$$+ \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta'} R(\zeta') = Z(\zeta) + \chi,$$

we have proves Equation (8.3.12). □

We notice here that from Propositions 8.3.3 and 8.3.4 we have recovered Equation (8.3.2).

Setting $v(\zeta) = v\zeta^{-n}$, $n \geq 0$, in (8.3.11) , where v is a constant $sl(N, \mathbb{C})$ matrix, we define

$$\chi^{(n)} = \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta'} Y(\zeta')vY(\zeta')^{-1}\zeta'^{-n}. \qquad (8.3.14)$$

From (8.3.13), $\chi^{(n)}$ satisfy the recursion relations (8.1.7). Then we can identify $\chi^{(n)}$ with $\lambda^{(n)}$ up to integration constants which arise from (8.1.8). Furthermore the infinitesimal transformation (8.3.11) with $\chi = \chi^{(n)}$ corresponds to (8.3.1), and Equation (8.3.12) with $\chi = \chi^{(n)}$ corresponds to (8.3.2). Now we conclude the hidden symmetry comes from the infinitesimal Riemann-Hilbert transformation.

In the remainder of this section we discuss the hidden symmetry algebra formed of the infinitesimal Riemann-Hilbert transformations. In accordance with Section 7.4, we consider a representation of the Transformation (8.3.10). Let us introduce an infinite number of potentials $\{G^{(m,n)}\}$, $m, n \in \mathbb{Z}$, by

$$\sum_{m,n=0}^{\infty} G^{(m,n)} \zeta^m \zeta'^n = \frac{1}{\zeta - \zeta'}(\zeta - \zeta' Y(\zeta)^{-1} Y(\zeta')),$$

and $G^{(m,-m)} = -G^{(-m,m)} = 1$ for $m \geq 1$, $G^{(m,n)} = 0$ for other negetive indices. By virtue of the recursion relations

$$G^{(m,n+1)} - G^{(m+1,n)} = G^{(m,1)} G^{(0,n)},$$

for $m, n \geq 0$, we see that the infinitesimal Riemann-Hilbert transformations associated with $v(\zeta) = v\zeta^{-k}$, $k \in \mathbb{Z}$, are represented as

$$G^{(m,n)} \to G^{(m,n)} + vG^{(m+k,n)} - G^{(m,n+k)}v$$
$$+ \sum_{j=1}^{k} G^{(m,j)} v G^{(k-j,n)}, \quad k \geq 0,$$

$$G^{(m,n)} \to G^{(m,n)} + vG^{(m+k,n)} - G^{(m,n+k)}v$$
$$+ \sum_{j=0}^{-k-1} \delta_{m,j} v \delta_{-k-j,n}, \quad k < 0, \quad (8.3.15)$$

for $m \geq 0$, $n \geq 1$. Denoting the transformation (8.3.15) by $v\zeta^{-k}$ for simplicity, we have the following theorem.

Theorem 8.3.5. *The commutation relations*

$$[\![v\zeta^{-k}, v'\zeta^{-l}]\!] = [v, v']\zeta^{-(k+l)} \qquad (8.3.16)$$

hold for and $v, v' \in sl(N, \mathbb{C})$ *and* $k, l \in \mathbb{Z}$.

The proof is carried out in the same way as in Theorem 7.4.4. As a corollary of this theorem, we claim that the hidden symmetry algebra of the infinitesimal Riemann-Hilbert transformations is isomorphic to

$$sl(N, \mathbb{C}) \otimes \mathbb{C}[\zeta, \zeta^{-1}].$$

For the solutions of the equations of motion, this algebra is regarded as the symmetry algebra of the $SL(N, \mathbb{C})$ chiral field equations.

Note that the Riemann-Hilbert transformation associated with $v\zeta^{-k}$ with $k < 0$ reduces to

$$Y'(\zeta) = Y(\zeta)\exp(-v\zeta^{-k})$$

which does not appear in the framework in [5, 6, 8]. This is the reason why they have derived only the subalgebra $g \otimes \mathbb{C}[\zeta]$ of $g \otimes \mathbb{C}[\zeta, \zeta^{-1}]$.

Finally, we discuss the cases of $G = SU(N)$ and $SO(N)$. For $SU(N)$ chiral fields, we further have to impose the additional constraints on $Y(\zeta)$ in (8.1.1) and $u(\zeta)$ in (8.3.3) such that

$$Y(\zeta)^{\dagger} Y(\zeta) = 1, \qquad u(\zeta)^{\dagger} u(\zeta) = 1,$$

where the daggers applied to $Y(\zeta)$ and $u(\zeta)$ indicate the hermitian conjugates of $Y(\zeta^*)$ and $u(\zeta^*)$, respectively. The resulting hidden symmetry algebra is $su(N) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$. On the other hand, for $SO(N)$ chiral fields, we impose

$$Y(\zeta)^{t} Y(\zeta) = 1, \qquad u(\zeta)^{t} u(\zeta) = 1.$$

Then we have the hidden symmetry algebra $so(N) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$. We note in addition that the equations of motion for $SU(2)$ chiral fields reduce to the well-known sine-Gordon equation (see Sections 2.1 and 2.2). It is concluded, from what has been shown above, that the infinite-dimensional Lie algebra $su(2) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ acts on the solution space of the sine-Gordon equation.

CHAPTER IX

CONCLUDING REMARKS

The main results obtained in the present thesis can be divided into two according to the following subjects;

(A) soliton theory of nonlinear field equations,

(B) symmetry theory of nonlinear fields.

Soliton theory is the subject of Chapters III, IV, and VI. An inverse scattering formula and a Bäcklund transformation for the stationary axially symmetric Einstein equations have been presented in Chapters III and IV. We have also discussed three types of Bäcklund transformations for the self-dual Yang-Mills equations in Chapter VI. These nonlinear equations are fundamental objects in the study of gravitational fields and gauge fields, respectively. We have derived concrete solutions in terms of the above Bäcklund transformations.

On the other hand, we have given a finite transformation of the Geroch group in gravitational fields to construct new family of exact solutions in Chapter V. The Ehlers-type symmetry and symmetry algebra for the self-dual Yang-Mills equations have been found in Chapter VII. Finally we have studied the Noether transformations of nonlinear fields in Chapter VIII. These discussions are mainly carried out on the subject mentioned in (B).

Although the soliton theory arises independently of the notion of symmetry, its significance will be understood to a good extent if we state their interrelation. We first mention the relationship between global symmetries and inverse scattering formulae. Here the global symmetry is a one such that it can be applied in the whole the space-time. A free parameter which appears in the inverse scattering formula reflects the existence of a global symmetry for the original nonlinear equation. For example, the Lie invariance of the Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$ under the Galilei transformation

$$x \to x + 6\lambda t, \quad t \to t, \quad u \to u + \lambda, \quad \lambda \in \mathbb{R}$$

brings a real parameter λ into the associated Schrödinger equation. Then we obtain one of the inverse scattering formula pair

$$Ly = (\frac{d^2}{dx^2} - u)y = \lambda y.$$

For the sine-Gordon equation $\omega_{xx} - \omega_{tt} = \sin \omega$, the Lorentz transformation

$$x \to (1 - v^2)^{-1/2}(x + vt),$$

$$t \to (1 - v^2)^{-1/2}(vx + t), \quad \omega \to \omega, \quad v \in \mathbb{R}$$

plays the same role. In Chapter III, a variable $\zeta(\xi, \eta)$ defined by

$$\zeta(\xi, \eta) = (\varepsilon + i\eta)^{1/2}(\varepsilon - i\xi)^{-1/2}, \quad \varepsilon \in \mathbb{R}$$

has emerged in the inverse scattering formula for the Einstein equations. This is due to the Lie invariance under a coodinate transformation [15]. In Chapter VI, we have introduced a set of fundamental variables

$$w, \quad w_1 = \bar{z} - wy, \quad w_2 = \bar{y} + wz, \quad w \in \mathbb{C}.$$

These serve as parameters corresponding to the full rotation group $SO(4)$ acting on the set of solutions of the self-dual equations [19]. Defining the differential operators

$$D_1 = w\partial_{\bar{z}} + \partial_y, \quad D_2 = w\partial_{\bar{y}} - \partial_z,$$

we have presented the inverse scattering formula.

We now turn to the derivation of an infinite number of nonlocal symmetries discussed in Chapter IV, VII, and VIII. Let the fundamental solution matrix $Y(\zeta)$ of the inverse scattering formulae be analytic near the origin $\zeta = 0$. Then each coefficient of Taylor expansion of $S(\zeta)$, defined by the formula $S(\zeta) = Y(\zeta)v(\zeta)Y(\zeta)^{-1}$, leads to a nonlocal symmetry of nonlinear field equations. Conversely, the generating function of nonlocal symmetries yields an inverse scattering formula. Hence, the existence of an infinite number of nonlocal symmetries is closely related to whether the soliton theory is applicable or not to the nonlinear field equations under consideration.

Futhermore, a symmetry called the Noether transformation is derived from one of the inverse scattering formula pair in Chapter VIII. This symmetry makes a change in the Lagrabgian density by a total divergence and, consequently, gives a weak continuity equation as a Noether current. Then under a suitable boundary condition, we obtain a constant of motion.

Next, we state the intersection of the subjects (A) and (B) in solution generating techniques. As a typical example, we discuss Geroch's transformation group in gravitational fields (see Section 5.1). A useful expression of infinitesimal transformations can be derived by using either internal symmetries of the field equations or the Riemann-Hilbert problem for the inverse scattering formula [11]. We then may regard the transformations that preserve asymptotic flatness [12, 13] as Bäcklund transformations. Conversely, the known Bäcklund transformations are recovered in the framework of the Geroch group [5].

As mentioned above, the notion of symmetry is relevant to the soliton theory of nonlinear field equations. In other words, the techniques in the soliton theory have been proposed through skillful treatments of the symmetries hidden in field equations. Hence, it is important to look for the full symmetries which the equations can admit, when we study more complicated and unsolved equations, such as the four-dimensional Einstein-(Maxwell) equations and the second-order Yang-Mills-(Higgs) equations.

In the preceding chapters, we have obtained some results in the study of soliton theory and symmetry of nonlinear field equations. However there are still a number of further problems in these directions which retain to be solved. We here list some of the problems;

(i) Hamiltonian structure,

(ii) twistor theory,

(iii) transformation group,

for nonlinear field equations.

The well-known Korteweg-de Vries equation can be written in the form

$$u_t = \frac{\partial}{\partial x}\frac{\delta H[u]}{\delta u}, \qquad H[u] = \int H\, dx$$

where $H = u^3 + \frac{1}{2}u_x$ and $\delta/\delta u$ is the Eular-Lagrange operator. It is proved that there is an infinite number of functionals $I_l[u] = \int R_l(u)dx$, $l \geq 1$, which are constant on the trajectories of the Hamiltonian flow corresponding to the Hamiltonian $H[u]$, [10]. Here $R_l(u)$ are polynomials in u and its derivatives of higher order. The functionals $I_l[u]$ are all in involution, that is, the Poisson brackets $\{I_l, I_m\}$ all vanish. This implies the complete integrability of the Korteweg-de Vries equation.

Though the first integrals of the chiral field equations have been given for the $SU(2)$ case [18], neither the Hamiltonian structure nor the involutiveness of first integrals for chiral field equations has yet been investigated. For example, we do not know what canonical variables are. Quite recently, Dickey [8] has discussed the Hamiltonian structure for the chiral fields.

The next outstanding problem is to make clear the relationship between the Atiyah-Ward theory of self-dual gauge fields [3] and the soliton theory developed in Chapter VI. The Atiyah-Ward theory is based on Penrose's twistor theory [17] which has an intimate connection with the differential geometry of four dimensions. By analytic continuation, we consider the self-dual Yang-Mills equations on $\mathbb{C}^4$. Let X be a point in $\mathbb{C}^4$ which we denote in the matrix form,

$$X = \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix}, \qquad (y, \bar{y}, z, \bar{z}) \in \mathbb{C}^4.$$

We introduce a plane β in $\mathbb{C}^4$ by

$$\beta = \{X \in \mathbb{C}^4 \mid \Omega = X\Pi, \ \Pi \neq 0\},$$

where Ω and Π are two-spinors. Since (Ω, Π) and $(\lambda\Omega, \lambda\Pi)$, $\lambda \in \mathbb{C}$, both define the same plane β, the planes in $\mathbb{C}^4$ are in one-one correspondence with the points of $\mathbb{P}^3$, the complex projective space of dimension three. The plane β and the space $\mathbb{P}^3$ are referred to as the anti-self-dual plane and the twistor space of the self-dual equations, respectively. Every 2-form which satisfies the self-dual equations should be annihilated on the plane β. We remark that our fundamental variables (w, w_1, w_2) are local coordinates in $\mathbb{P}^3$. By breaking $T(\zeta)$ into $T(\zeta) = Y^{(+)}(\zeta)^{-1}Y^{(-)}(\zeta)$, we have shown that the matrices $Y^{(\pm)}(\zeta)$ give rise to self-dual gauge potentials in Section 6.4. Atiyah and Ward claimed that in order to construct the instanton solutions it is sufficient to take $T(\zeta)$ as

$$T(\zeta) = \begin{pmatrix} \zeta^{-N} & \psi(X,\zeta) \\ 0 & \zeta^{N} \end{pmatrix}, \qquad D_k\psi(X,\zeta) = 0.$$

Since $Y^{(\pm)}(\zeta)$ are the fundamental solutions of the linear system whose compatibility condition is equivalent to the self-dual equations, the transition matrix $T(\zeta)$ is analogous to the scattering matrix appearing in the usual inverse scattering method. In this sense, the Atiyah-Ward theory is therefore closely related to the soliton theory.

As is well known, the inverse scattering method for solving two-dimensional equations can be considered a natural extension of Fourier transform [1]. The transform

$X \to (\Omega, \Pi)$ defined by $\Omega = X\Pi$ may also be compared with the Fourier transform [2]. The twistor theory has been successfully applied to other four-dimensional equations such as the self-dual Einstein equations [16].

Finally, we make a brief mention of the transformation group acting on solutions to nonlinear field equations, especially to the self-dual Yang-Mills equations. Even the Lie algebra of the infinitesimal transformations reflect the feature of the solution space to some extent. In Chapter VII, it was proved that the infinite-dimensional Lie algebra $gl(n, \mathbb{C}) \otimes \mathbb{C}[w, w^{-1}, w_1, w_2]$ acts on the solution space of the $GL(n, \mathbb{C})$ self-dual Yang-Mills equations $\partial_y(\partial_{\bar{y}}P\cdot P^{-1}) + \partial_z(\partial_{\bar{z}}P\cdot P^{-1}) = 0$, $P \in GL(n, \mathbb{C})$, as a symmetry algebra. We recall that the Lie algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}[\zeta, \zeta^{-1}]$ of the Geroch group acts transitively on the solution space of the stationary axially symmetric Einstein equations. Furthermore, the algebras $su(n) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$ and $su(2) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$ act on the $SU(n)$ chiral fields [21] and the Heisenberg spin model [9], respectively.

Apart from these works, Sato [20] pointed out that the solution space of the Kadomtsev-Petviashivili (KP) hierarchy has a structure of infinite-dimensional Grassmann manifold. He also conjectured that every two-dimensional soliton equation could be derived by a suitable reduction of the KP hierarchy. The automorphism group $GL(\infty)$ of the Grassmann manifold may be accepted as the transformation group of the KP heirarchy. Indeed, using infinite-order operators, Date, Jimbo, Kashiwara, and Miwa [6, 7] claimed that the Lie algebra $gl(\infty)$ acts on the solution space. Consequently, an affine Lie algebra which acts on each soliton equation can be given as a Lie subalgebra of $gl(\infty)$. For example, the infinitesimal transformation group of the Korteweg-de Vries equation is the affine Lie algebra $A_1^{(1)}$, a central extension of $sl(2, \mathbb{C}) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$. It is, however, hard to apply this procedure to three and four-dimensional equations such as the self-dual Yang-Mills equations and to systems with finite degrees of freedom such as the finite nonperiodic Toda lattice.

The self-dual Yang-Mills equations (SDE) are, so to speak, the most general of relativistic invariant field equations solvable in the framework of soliton theory because of the following reasons. The chiral field equations are equivalent to the two-dimensional SDE such that $y = \bar{y}$ and $z = \bar{z}$. The static axially symmetric $SU(2)$ SDE are reduced to the stationary axially symmetric Einstein equations in gravitational fields [23]. The static SDE can be identified with the Bogomolny equations [4] in three-dimensional Yang-Mills-Higgs fields. Moreover, the solutions of the $SU(2)$ SDE which are invariant under three-dimensional rotations can be given by solving the Liouville equation [22]. Other spherically symmetric SDE are reduced to the two-dimensional finite nonperiodic Toda lattice equations [14]. Thus the complete understanding of transformation groups acting on the solution space of the SDE, if possible, will give us a wider outlook on the theory of nonlinear field equations. We expect that the symmetry algebra $gl(n, \mathbb{C}) \otimes \mathbb{C}[w, w^{-1}, w_1, w_2]$ plays a key role to studying the transformation group.

ACKNOWLEDGEMENTS

The author would like to express his sincere thanks to Professor Mineo IKEDA at Kyoto University for his carefully reading the manuscript and offering workable comments. He has pursued this research under the proper guidance and encouragement of M. IKEDA. He is grateful to Professor Yujiro Ohya at Kyoto University for his valuable suggestions and comments. Thanks are also due to Professor Chiaki IHARA at Kyoto University for giving him encouragement in carrying out this work. The author especially wishes to express his gratitude to Assistant Professor Toshihiro IWAI at Kyoto University, who read the manuscript and proffered helpful advices.

He is indebted to Assistant Professor Junkichi SATSUMA at Miyazaki Medical College and Assistant Professor Ruishi KUWABARA at Tokushima University for informative conversations on this research. It is also pleasure to thank Lecturer Kimio UENO at Yokohama City University for fruitful discussions in collaborations.

REFERENCES

CHAPTER I

[1] Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. Lond. A 362, 425-461(1978)

[2] Atiyah, M.F., Ward, R.S.: Instantons and algebraic geometry. Commun. Math. Phys. 55, 117-124(1977)

[3] Belavin, A.A., Polyakov, A.M., Schwartz, A.S., Tyupkin, Yu.S.: Pseudoparticle solutions of the Yang-Mills equations. Phys. Lett. 59B, 85-87(1975)

[4] Belavin, A.A., Zakharov, V.E.: Yang-Mills equations as inverse scattering problem. Phys. Lett. 73B, 53-57(1978)

[5] Belinski, V.A., Zakharov, V.E.: Stationary gravitational solitons with axial symmetry. Sov. Phys. JETP 70, 1-9(1979)

[6] Corrigan, E.F., Fairlie, D.B., Yates, R.G., Goddard, P.: The construction of self-dual solutions to $SU(2)$ gauge theory. Commun. Math. Phys. 58, 223-240 (1978)

REF. CHAP. I

[7] Dolan, L.: Kac-Moody algebra is hidden symmetry of chiral models. Phys. Rev. Lett. 47, 1371-1374(1981)

[8] Dolan, L., Roos, A.: Nonlocal currents as Noether currents. Phys. Rev. D 22, 2018-2021(1980)

[9] Eisenhart, L.P.: A treatise on the differential geometry of curves and surfaces. New York: Dover 1960

[10] Ernst, F.J.: New formulation of the axially symmetric gravitational field problem. Phys. Rev. 167, 1175-1178(1968)

[11] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: A method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. 19, 1095-1097(1967)

[12] Geroch, R.: A method for generating new solutions of Einstein's equation, II. J. Math. Phys. 13, 394-404(1972)

[13] Harrison, B.K.: Bäcklund transformation for the Ernst equation of general relativity. Phys. Rev. Lett. 41, 1197-1200(1978)

[14] Hauser, I., Ernst, F.J.: Integral equation method for effecting Kinnersley-Chitre transformations.

Ref. Chap. I

Phys. Rev. D20, 362-369(1979)

[15] Hauser, I., Ernst, F.J.: A homogeneous Hilbert problem for the Kinnersley-Chitre transformations. J. Math. Phys. 21, 1126-1140(1980)

[16] Hauser, I., Ernst, F.J.: Proof of a Geroch conjecture. J. Math. Phys. 22, 1051-1063(1981)

[17] Ikeda, M., Maekawa, T., Miyachi, Y.: On the form of spherically symmetric Yang-Mills fields. Math. Japon. 28, 143-147(1983)

[18] Ikeda, M., Miyachi, Y.: On the static and spherically symmetric solutions of the Yang-Mills fields. Prog. Theor. Phys. 27, 474-482(1962)

[19] Jaffe, A., Taubes, C.: Vortics and monopoles, structure of static gauge theories. Boston, Basel, Stuttgart: Birkhäuser 1980

[20] Kinnersley, W.: Symmetries of the stationary Einstein-Maxwell field equations, I. J. Math. Phys. 18, 1529-1537(1977)

[21] Kinnersley, W., Chitre, D.M.: Symmetries of the stationary Einstein-Maxwell field equations, II. J. Math. Phys. 18, 1538-1542(1977)

Ref. Chap. I

[22] Kinnersley, W., Chitre, D.M.: Symmetries of the stationary Einstein-Maxwell field equations, III. J. Math. Phys. 19, 1926-1931(1978)

[23] Kinnersley, W., Chitre, D.M.: Symmetries of the stationary Einstein-Maxwell field equations, IV. Transformation which preserve asymptotic flatness. J. Math. Phys. 19, 2037-2042(1978)

[24] Kramer, D., Stephani. H., Herlt, E., MacCallum, M.: Exact solutions of Einstein's field equations. Cambridge monographs on mathematical physics. Cambridge U. P. 1981

[25] Lamb, Jr., G.L.: Analytical description of ultra-short optical pulse propagation in a resonant medium. Rev. Mod. Phys. 43, 99-124(1971)

[26] Lepowsky, J., Wilson, R.L.: Construction of the affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. 62, 43-53(1978)

[27] Lund, F.: Note on the geometry of the nonlinear σ-model in two dimensions. Phys. Rev. D 15, 1540-1543 (1977)

[28] Lüsher, M., Pohlmeyer, K.: Scattering of massless

Ref. Chap. I

lumps and non-local charges in the two dimensional classical nonlinear σ-model. Nucl. Phys. B 137, 46-54(1978)

[29] Maison, D.: On the complete integrability of the stationary, axially symmetric Einstein equations. J. Math. Phys. 20, 871-877(1979)

[30] Miura, R.M., Gardner, C.S., Kruskal, M.D.: The Korteweg-de Vries equation and generalizations, II. Existence of conservation laws and constants of motion. J. Math. Phys. 9, 1204-1209(1968)

[31] Miyachi, Y., Ikeda, M., Maekawa, T.: On the spherically symmetric solutions of the classical $SU(2)$ Yang-Mills fields. Prog. Theor. Phys. 68, 261-276 (1982)

[32] Neugebauer, G.: Bäcklund transformations of axially symmetric stationary gravitational fields. J. Phys. A12, L67-L70(1979)

[33] Penrose, R.: The twistor programme. Rep. Math. Phys. 12, 65-76(1977)

[34] Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through quadratic constraints. Commun. Math. Phys. 46, 207-221(1976)

REF. CHAP. I

[35] Pohlmeyer, K.: On the Lagrangian theory of anti-self-dual fields in four-dimensional Euclidean space. Commun. Math. Phys. 72, 37-47(1980)

[36] Prasad, M.K., Sinha, A., Wang, L-L.C.: Parametric Bäcklund transformation for self-dual *SU*(N) Yang-Mills fields. Phys. Rev. Lett. 43, 750-753(1979)

[37] Prasad, M.K., Sinha, A., Wang, L-L.C.: Non-local continuity equations for self-dual *SU*(N) Yang-Mills fields. Phys. Lett. 87B, 237-238(1979)

[38] Ueno. K.: Infinite dimensional Lie algebras acting on chiral fields and the Riemann-Hilbert problem. Publ. RIMS, Kyoto Univ. 19, 59-82 (1983)

[39] Wahlquist, H.D., Estabrook, F.B.: Prolongation structure of nonlinear evolution equations. J. Math. Phys. 16, 1-7(1975)

[40] Ward, R.S.: On self-dual gauge fields. Phys. Lett. 61A, 81-82(1977)

[41] Yang, C.N.: Construction of self-duality for *SU*(2) gauge fields on Euclidean four-dimensional space. Phys. Rev. Lett. 38, 1377-1379(1977)

Ref. Chap. I

[42] Zakharov, V.E., Faddeev, L.D.: Korteweg-de Vries equations, a complete integrable Hamiltonian system. Func. Anal. Appl. 5, 280-287(1971)

[43] Zakharov, V.E., Mikhailov, A.V.: Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Sov. Phys. JETP 47, 1017-1027(1979)

[44] Zakharov, V.E., Shabat, A.B.: A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, I. Func. Anal. Appl. 8, 226-235(1974)

CHAPTER II

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: The inverse scattering transform - Fourier analysis for nonlinear problems. Stud. Appl. Math. 53, 249-315(1974)

[2] Eisenhart, L.P.: see I. [9]

[3] Faddeev, L.D., Korepin, V.E.: Quantum theory of

REF. CHAP. II

soliton. Phys. Rep. 42, 1-87(1978)

[4] Lamb, Jr., G.L.: see I. [25]

[5] Lüsher, M., Pohlmeyer, K.: see I. [28]

[6] Perelomov, A.M.: Instanton-like solutions in chiral models. Physica 1D, 1-25(1981)

[7] Pohlmeyer, K.: see I. [34]

[8] Scott, A.C.: Josephson transmission line osillator. In: Soliton and condensed matter physics. Bishop, A.R., Schneider, T. Eds. Berlin-Heidelberg-New York: Springer 1978

[9] Wahlquist, H.D., Estabrook, F.B.: see I. [39]

[10] Zakharov, V.E., Mikhailov, A.V.: see I. [43]

[11] Zakharov, V.E., Shabat, A.B.: Integration of nonlinear equations of mathematical physics by the method of inverse scattering, II. Func. Anal. Appl. 13, 166-174(1979)

CHAPTER III

[1] Belinski, A.A., Zakharov, V.E.: see I. [5]

[2] Ernst, F.J.: see I. [10]

[3] Flanders, H.: Differential forms with application to the physical science. New York-London: Academic Press 1963

[4] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: see I. [11]

[5] Harrison, B.K.: see I. [13]

[6] Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. Commun. Pura Appl. Math. 21, 467-490(1968)

[7] Maison, D.: see I. [29]

[8] Papapetrou, V.A.: Eine rotationssymmetrische Lösung in der allgemeinen Relativitätstheorie. Ann. Physik Leipzig. 6. F. 12, 309-315(1953)

[9] Wahlquist, H.D., Estabrook, F.B.: see I. [39]

[10] Zakharov, V.E., Shabat, F.B.: see I. [44]

CHAPTER IV

[1] Belinsky, V.A., Zakharov, V.E.: see I. [5]

[2] Harrison, B.K.: see I. [13]

[3] Hauser, I., Ernst, F.J.: see I. [15]

[4] Kasner, E.: Solutions of the Einstein equations involving functions of only one variable. Tran. Am. Math. Soc., 27, 155-162(1925)

[5] Kramer, D., Stephani, H., Herlt, E., MacCallam, M.: see I. [24]

[6] Lamb, Jr., G.L.: see I. [25]

[7] Neugebauer, G.: see I. [32]

[8] Ward, R.S.: A Yang-Mills-Higgs monopole of charge 2. Commun. Math. Phys. 79, 317-325(1981)

[9] Zakharov, V.E., Mikhailov, A.V.: see I. [43]

CHAPTER V

[1] Atiyah, M.F., Ward, R.S.: see I. [2]

[2] Belinski, V.A., Zakharov, V.E.: see I. [5]

[3] Corrigan, E.F., Fairlie, D.B., Yates, R.G., Goddard,

Ref. Chap. V

P.: see I. [6]

[4] Cosgrove, C.M.: Relationship between the group-theoretic and soliton-theoretic techniques for generating stationary axisymmetric gravitational solutions. J. Math. Phys. 21, 2417-2477(1980)

[5] Ehlers, J.: Les theories de la gravitation. Paris: CNRS 1959

[6] Geroch, R.: see I. [12]

[7] Harrison, B.K.: see I. [13]

[8] Hoenselaers, C., Kinnersley, W., Xanthopoulos, C.: Symmetries of the stationary Einstein-Maxwell equations, VI. Transformations which generate asymptotically flat spacetimes with arbitrary multipole moments. J. Math. Phys. 20, 2530-2536 (1979)

[9] Kac, V.G.: Simple irreducible graded Lie algebras of finite growth. Izv. Akad. Nauk. USSR 32, 1323-1367(1968)

[10] Kinnersley, W.: Recent progress in exact solutions. In: General relativity and gravitation. Shaviv, G., Rosen, J. Eds. New York: Wiley 1975

REF. CHAP. V

[11] Kinnersley, W.: see I. [20]

[12] Kinnersley, W., Chitre, D.M.: see I. [21]

[13] Kinnersley, W., Chitre, D.M.: see I. [22]

[14] Kinnersley, W., Chitre, D.M.: see I. [23]

[15] Moody, R.: A new class of Lie algebra. J. Alge. 10, 211-230(1968)

[16] Neugebauer, G.: see I. [32]

[17] Neugebauer, G.: A general integral of the axially symmetric stationary Einstein equations. J. Phys. A: Math. Gen., L19-L21(1980)

[18] Neugebauer, V.G., Kramer, D.: Eine Methods zur Konstruktion stationärer Einstein-Maxwell-Felfer. Ann. Physik 24, 62-71(1969)

CHAPTER VI

[1] Atiyah, M.F., Hitchin, N.J., Drienfeld, V.G., Manin, Yu.I.: Construction of instantons. Phys. Lett. 65A, 185-187(1978)

Ref. Chap. VI

[2] Atiyah, M.F., Ward, R.S.: see I. [2]

[3] Belavin, A.A., Zakharov, V.E.: see I. [4]

[4] Belinski, V.A., Zakharov, V.E.: see I. [5]

[5] Brihaya, Y., Fairlie, D.B., Nuts, J., Yates, R.G.: Properties of the self dual equations for an *SU*(n) gauge theory. J. Math. Phys. 19, 2528-2532(1978)

[6] Corrigan, E.F., Fairlie, D.B., Yates, R.G., Goddard, P.: see I. [6]

[7] Gantmacher, F.B.: The theory of matrices. New York: Chelsea 1959

[8] Pohlmeyer, K.: see I. [35]

[9] Prasad, M.K.: Instanton and monopoles in Yang-Mills gauge field theories. Physics 1D, 167-191(1980)

[10] Prasad, M.K., Sinha, A., Wang, L-L.C.: see I. [36]

[11] Ward, R.S.: see I. [40]

[12] Yang, C.N.: see I. [41]

[13] Yang, C.N., Mills, R.L.: Conservation of isotopic spin and isotopic gauge invariance. Phys. Rev. 96, 191-195(1954)

[14] Zakharov, V.E., Mikhailov, A.V.: see I. [43]

CHAPTER VII

[1] Chau, L.-L., Ge, M.-L., Wu, Y.-S.: Kac-Moody algebra in the self-dual Yang-Mills equation. Phys. Rev. D 25, 1086-1095(1982)

[2] Dolan, L.: see I. [7]

[3] Ehlers, J.: see V. [5]

[4] Forgács, P., Horváth, Z., Palla, L.: Towards complete integrability of the self-duality equations. Phys. Rev. D 23, 1876-1879(1981)

[5] Geroch, R.: see I. [12]

[6] Hauser, I., Ernst, F.J.: see I. [15]

[7] Kinnersley, W.: see I. [20]

[8] Kinnersley, W., Chitre, D.M.: see I. [21]

[9] Maison, D.: Ehlers-Harrison-type transformations for Jordan's extended theory of gravitation. Gen. Rel. Grav. 10, 717-723(1979)

[10] Muskhelishvili, N.J.: Singular integral equations. Leyden: Noordhoff 1958

[11] Pohlmeyer, K.: see I. [34]

[12] Ueno, K.: see I. [38]

[13] Yang, C.N.: see I. [41]

CHAPTER VIII

[1] Bogomol'nyi, E.B.: The stability of classical solutions. Sov. J. Nucl. Phys. 24, 449-454(1976)

[2] Brézin, E., Itzykson, C., Zinn-Justin, J., Zuber, J.-B.: Remarks about the existence of nonlocal charges in two-dimensional models. Phys. Lett. 82B, 442-444(1979)

[3] Candotti, E., Palmieri, C., Vitale, B.: On the inversion of Noether's theorem in the Lagrangian formalism. II. - Classical field theory. Nuovo. Cimento A70, 233-246(1970)

[4] Chodos, A.: Conserved quantities in classical Yang-Mills theory. Phys. Rev. D 20, 915-920(1979)

[5] Devchand, C., Fairlie, D.B.,: A generating function for hidden symmetries of chiral models. Nucl. Phys. B 194, 232-236(1982)

[6] Dolan, L.: see I. [7]

[7] Dolan, L., Roos, A.: see I. [8]

[8] Ge, M.-L., Wu, Y.-S.: An explicit approach to the group structure of hidden symmetry. Phys. Lett. 108B, 411-415(1982)

[9] 't Hooft, G.: Magnetic monopoles in unified gauge

Ref. Chap. VIII

theory. Nucl. Phys. B 79, 276-284(1974)

[10] Hou, B., Ge, M.-L., Wu, Y.-S.: Noether analysis for the hidden symmetry responsible for an infinite set of nonlocal currents. Phys. Rev. D 24, 2238-2244 (1981)

[11] Muskhelishivili, N.J.: see VII. [10]

[12] Prasad, M.K., Sommerfields, C.M.: Exact classical solution for the 't Hooft monopole and Julia-Zee dyon. Phys. Rev. Lett. 35, 760-762(1975)

[13] Rosen, J.: Noether's theorem in classical field theory. Ann. Phys. 69, 349-363(1972)

[14] Ueno, K.: see I. [38]

[15] Ward, R.S.: see IV. [8]

Chapter IX

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: see II. [1]

[2] Atiyah, M.F.: Real and complex geometry in four-dimensions. In: The Chern symposium 1979. New York,

REF. CHAP. IX

Heidelberg, Berlin: Springer 1980

[3] Atiyah, M.F., Ward, R.S.: see I. [2]

[4] Bogomol'nyi, E.B.: see VIII. [1]

[5] Cosgrove, C.M.: see V. [4]

[6] Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Operator approach to Kadomtsev-Petviashivili equation. J. Phys. Soc. Jpn. 50, 3806-3812(1981)

[7] Date, E., Kashiwara, M., Miwa, T.: Vertex operators and τ-functions. Proc. Japan. Acad. 57A, 387-392(1981)

[8] Dickey, L.A.: Symplectic structure, Lagrangian, and involutiveness of first integrals of the principal chiral field equation. Commun. Math. Phys. 87, 505-513(1983)

[9] Eichenherr, H.: Symmetry algebra of the Heisenberg-model and the non-linear Schrödinger equation. Phys. Lett. 115B, 385-388(1982)

[10] Gel'fand, I.M., Dikii, L.A.: Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation. Russ. Math. Survey 30, 77-113(1975)

Ref. Chap. IX

[11] Hauser, I., Ernst, F.J.: see I. [15]

[12] Hoenselars, C., Kinnersley, W., Xanthopoulos, C.: see V. [8]

[13] Kinnersley, W., Chitre, D.M.: see I. [23]

[14] Leznov, A.V., Saveliev, M.V.: Representation theory and integration of nonlinear spherically symmetric equations to gauge theories. Commun. Math. Phys. 74, 111-118(1980)

[15] Papanicolaou, N.: Gravitational duality and Bäcklund transformations. J. Math. Phys. 20, 2069-2077(1979)

[16] Penrose, R.: Nonlinear gravitons and curved twistor theory. Gen. Rel. Grav. 7, 31-52(1976)

[17] Penrose, R.: see L. [33]

[18] Pohlmeyer, K.: see I. [34]

[19] Pohlmeyer, K.: see I. [35]

[20] Sato, M.: Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold. In: Random system and dynamical system. RIMS Kokyuroku 439. 1981

Ref. Chap. IX

[21] Ueno, K.: see I. [38]

[22] Witten, E.: Some exact multipseudoparticle solutions of classical Yang-Mills theory. Phys. Rev. Lett. 38, 121-124(1977)

[23] Witten, L.: Static axially symmetric solutions of self-dual $SU(2)$ gauge fields in Euclidean four-dimensional space. Phys. Rev. D 19, 718-720(1979)

LIST OF AUTHOR'S PAPERS CITED IN THIS THESIS

[1] Nakamura, Y.: On inverse scattering formula for Ernst equation. Math. Japon. 24, 469-472(1979)

[2] Nakamura, Y.: On the construction of exact solutions of the stationary axially symmetric Einstein equations. Ann. Mat. Pura Appl. 129, 153-160(1981)

[3] Nakamura, Y.: Symmetries of stationary axially symmetric vacuum Einstein equations and the new family of exact solutions. J. Math. Phys. 24, 606-609(1983)

[4] Nakamura, Y.: Ehlers-type transformation for $SL(2, \mathbb{C})$ self-dual equations of gauge theory. to appear in Lett. Math. Phys.

[5] Ueno, K., Nakamura, Y.: Transformation theory for anti-self-dual equations and the Riemann-Hilbert problem. Phys. Lett. 109B, 273-278(1982)

[6] Ueno, K., Nakamura, Y.: The hidden symmetry of chiral fields and the Riemann-Hilbert problem. Phys. Lett. 117B, 208-212(1982)

[7] Ueno, K., Nakamura, Y.: Transformation theory for anti-self-dual equations. to appear in Publ. Res. Inst. Math. Sci., Kyoto Univ.